Indian Institute of Management Calcutta

Working Paper Series

WPS No. 787
September 2016

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Improving DEA efficiency under constant sum of inputs/outputs and common weights

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Abstract

In Data Envelopment Analysis (DEA), one way of calculating efficiency is to use parameter weights common to all decision-making units (DMUs), since it is reasonable for similar DMUs to place similar weights on inputs and outputs. At the same time, in many situations, the total amount of input or output available to a set of DMUs is fixed. In this paper, we have formulated DEA models to calculate the best strategy for improving the efficiency of an inefficient DMU when the parameters of all DMUs are weighted with a common set of weights, and there is a constant sum of inputs/outputs constraint. Theoretical results have been illustrated with the help of a numerical example.

Keywords: DEA, efficiency, linear programming, common weights, constant sum of inputs, constant sum of outputs

1 Introduction

Data Envelopment Analysis (DEA) is a widely applied non-parametric mathematical programming technique to calculate the relative efficiency of firms/organizations or Decision Making Units (DMUs) operating in a similar environment and utilizing multiple inputs to produce multiple outputs. Based on Farrell's (1957) work on productive efficiency, DEA was first introduced by Charnes, Cooper, and Rhodes (1978). Efficiency obtained using DEA, in its simplest form, is the comparison of the weighted output to weighted input ratio of the observing DMU with that of the best practice in the group. DEA has an advantage over other methods since the inputs/output weights are determined by the DEA model itself, and thus the decision maker does not face the problem of determining the weights beforehand. Measurement of efficiency is important to shareholders, managers, and investors for any future course of action. DEA has been extensively applied to a wide spectrum of practical problems. Examples include financial institutions (Sherman and Ladino 1995), bank failure prediction (Barr et. al. 1994), electric utilities evaluation (Goto and Tsutsui 1998), textile industry performance (Chandra et. al. 1998), sports (Singh and Adhikari, 2015; Singh 2011; Ruiz et al. 2013), portfolio evaluation (Murthi et. al. 1997). DEA analysis makes allowance for DMUs which are under constant returns to scale (CRS) using the Charnes-Cooper-Rhodes (CCR) models (1978) as well as for variable returns to scale (VRS) models such as the Banker-Charnes-Cooper (BCC) models (Banker et al. 1984).

A DMU can improve its efficiency by reducing its input or increasing its output or doing both. However, when a DMU attempts to increase output, it may turn out that there is only a limited amount of output that can be produced in the system by all DMUs, such as the Olympic Games where only a limited number of medals are available to the competing teams (Lins et. al. 2003), or competition for market share (Hu and Fang 2010). This limitation on output is called Constant Sum of Outputs (CSOO). Yang et. al. (2011) established models for improving the efficiency of a DMU under CSOO, but these models all operated under the assumption that the various inputs and outputs of a DMU were freely substitutable, and that the weights of one DMU were independent of the weights for other DMUs.

Similarly, if the amount of input in the system is constant, a DMU can only reduce its input if the input of another DMU is increased. Examples of constant sum of input (CSOI) problems include the distribution of a fixed cost (Cook and Kress 1999) or the allocation of office space (Gomes et al. 2008). Almost all approaches to the CSOI problem such as those developed by Beasley (2003) or later methods like the ellipsoidal frontier model (Milioni et al. 2011), treat the CSOI as a fixed cost problem. Lotfi et al. (2013) deals with resource allocation under common weights, but it also treats the problem as a fixed cost allocation. None of them consider the problem of a single DMU attempting to improve its efficiency, instead looking at all DMUs simultaneously. Furthermore, no paper exists that combines CSOI and CSOO constraints in a single problem.

The survey of existing works in CSOO and CSOI indicate that there are few papers that take into account the Common Weights restriction, and none of these addresses both CSOO and CSOI problems simultaneously. Common Weights (CW) is a very common form of weight restriction in DEA. If the DMUs are operating in a similar environment, they can be expected to have the same parameter weights (Lotfi et al. 2013). The CW approach is also used when classic DEA models do not give us accurate information as to the real weights of the input and output parameters, or to differentiate between efficient DMUs (Liu and Peng 2008). Several approaches exist for solving CW efficiency (Roll and Golany 1993, Cook and Zhu 2007). In this paper, we have addressed the problem of efficiency improvement under CSOI and CSOO constraints when a set of common weights is applied to input and output parameters of all DMUs. This is useful in real-life situations where the overall efficiency of the system is more important than individual DMU's efficiency (Beasley 2003), and there are limits to total productivity and/or a fixed input quantity. It is also useful in situations where all DMUs are expected to place similar weights on the various parameters.

This paper is organized as follows. In section 2, DEA models and theoretical results have been developed. Numerical illustration of the theoretical results developed in this paper is given in section 3. Conclusion and future direction for research have been presented in section 4.

2 Model formulation and theoretical results

The following notations have been used throughout this paper. Other notations used in certain sections will be defined at appropriate places.

n : The number of DMUs.
m : The number of inputs.
s : The number of outputs.

x_{ij} : Observed amount of j^{th} input for the i^{th} DMU ($i = 1, \ldots, n; j = 1, \ldots, m$).
y_{ir} : Observed amount of r^{th} output for the i^{th} DMU($i = 1, \ldots, n; r = 1, \ldots, s$).
θ_k^{CW} : E□ciency of the k^{th} DMU under common weights (CW).
u_r : The weight assigned to the r^{th} output($r = 1, \ldots, s$).
v_j : The weight assigned to the j^{th} input($j = 1, \ldots, m$).
u_0 : Value representing the variable part of variable returns to scale DEA models.
Δ_i^I : Virtual gap between the weighted inputs of the $i^{th}(i = 1, \ldots, n)$ DMU and best-practices frontier.
Δ_i^O : Virtual gap between the weighted outputs of the $i^{th}(i = 1, \ldots, n)$ DMU and best-practices frontier.
g_{kr} : The output increase in the $r^{th}(r = 1, \ldots, s)$ output of the k^{th} DMU.
t_{ir} : The output reduction in the $r^{th}(r = 1, \ldots, s)$ output of the $i^{th}(i = 1, \ldots, n; i = k)$ DMU.
f_{kj} : The input decrease in the $j^{th}(j = 1, \ldots, m)$ input of the k^{th} DMU.
s_{ij} : The input increase in the $j^{th}(j = 1, \ldots, m)$ input of the $i^{th}(i = 1, \ldots, n; i = k)$ DMU.
: An infinitesimally small positive value.

Several methods exist for ranking DMUs' e□ciency under common weights (Roll and Golany 1993, Cook and Zhu 2007, Liu and Peng 2008, Lotfi et al. 2013). Under the CW limitation, since the weights are same across all DMUs, the objective is to select weights such that the overall e□ciency of all DMUs is maximized. Later papers use a Goal Programming (Tamiz et al. 1998) approach to convert the problem into a s30(co424(e)-1(t)-424(a)1(l.)-729(19)1(98))-424(a)1(pproa)1(c)27(h)-425(t)1(o)-424(con)27(v) model to achieve this minimization is shown below.

$$(M1) \quad Min \sum_{i=1}^{n} (\Delta_i^O + \Delta_i^I)$$

subject to

$$\frac{\sum_{r=1}^{s} u_r y_{ir} + \Delta_i^O}{\sum_{j=1}^{m} v_j x_{ij} - \Delta_i^I} = 1, i = 1, \ldots, n,$$

$$v_j, u_r, \Delta_i^O, \Delta_i^I \geq 0; \ i = 1, \ldots, n; \ j = 1, \ldots, m; \ r = 1, \ldots, s.$$

Model (

Now, define the values of Δ_i^O and Δ_i^I $(i = k)$ as

$$\Delta_i^O = \Delta_i^{O*} + \sum_{r=1}^{d} u_r^* t_{ir}, \; (i = 1, \ldots, n; i = k) \tag{2.5}$$

$$\Delta_i^I = \Delta_i^{I*} + \sum_{j=1}^{c} v_j^* s_{ij}, \; (i = 1, \ldots, n; i = k) \tag{2.6}$$

Since $(M2)$ has been solved, and it is equivalent to $(M1)$, all the constraints of $(M1)$ are fulfilled by its solution. Thus,

$$\frac{\sum_{r=1}^{s} u_r^* y_{ir} + \Delta_i^{O*}}{\sum_{j=1}^{m} v_j^* x_{ij} - \Delta_i^{I*}} = 1, \; i = 1, \ldots, n.$$

The above equation can be rewritten using eqns. (2.1)-(2.6) as

$$\frac{\sum_{r=1}^{s} u_r^* y_{kr} + \sum_{r=1}^{d} u_r^* g_{kr}}{\sum_{j=1}^{m} v_j^* x_{kj} - \sum_{j=1}^{c} v_j^* f_{kj}} = 1, \tag{2.7}$$

$$\frac{\sum_{r=1}^{s} u_r^* y_{ir} + \Delta_i^O - \sum_{r=1}^{d} u_r^* t_{ir}}{\sum_{j=1}^{m} v_j^* x_{ij} - \Delta_i^I + \sum_{j=1}^{c} v_j^* s_{ij}} = 1, \; i = 1, \ldots, n; i = k, \tag{2.8}$$

$$f_{kj} = \sum_{i=1, i \neq k}^{n} s_{ij}, \; j = 1, \ldots, c, \tag{2.9}$$

$$g_{kr} = \sum_{i=1, i \neq k}^{n} t_{ir}, \; r = 1, \ldots, d, \tag{2.10}$$

$$0 \le t_{ir} \le y_{ir} - \;\;, \; r = 1, \ldots, d; \; i = 1, \ldots, n; \; i = k, \tag{2.11}$$

$$0 \le f_{kj} \le x_{kj} - \;\;, \; j = 1, \ldots, c; \; i = 1, \ldots, n; \; i = k, \tag{2.12}$$

The equations (2.7)-(2.12) are identical to the constraints of model $(M3)$, and they all hold good as they are equivalent to the constraints of $(M1)$. Thus the values of u_r^*, v_j^*, t_{ir}, s_{ij}, Δ_i^O, Δ_i^I as defined by $(M2)$ and eqns. (2.1)-(2.6) represent a feasible solution to model $(M3)$. Hence the proof. ■

Let E^* be the overall efficiency of all DMUs as calculated by $(M2)$, and E be overall efficiency of all DMUs using the model $(M3)$.

Theorem 2. $E \ge E^*$ for all solutions of $(M3)$.

Proof. The values of u_r^*, v_j^*, t_{ir}, s_{ij}, Δ_i^O, Δ_i^I as defined in Theorem 1, represent a feasible solution to

By the constraints of $(M4)$,

$$\Delta_i^O, \Delta_i^I \geq 0,$$

which means

$$\sum_{r=1}^{s} u_r y_{ir} - \sum_{j=1}^{m} v_j x_{ij} + \Delta_i^O + \Delta_i^I - \sum_{r=1}^{d} \varphi_{ir} - \sum_{j=1}^{c} y_{kj} = 0,\ i = 1, \ldots, n;\ i \neq k,$$

can be rewritten as

$$\sum_{j=1}^{m} v_j x_{ij} - \sum_{r=1}^{s} u_r y_{ir} + \sum_{r=1}^{d} \varphi_{ir} + \sum_{j=1}^{c} y_{kj} \geq 0,\ i = 1, \ldots, n;\ i \neq k. \tag{2.13}$$

Also by the constraints of $(M4)$, since

$$\Delta_i^I + \Delta_i^O = \sum_{j=1}^{m} v_j x_{ij} - \sum_{r=1}^{s} u_r y_{ir} + \sum_{r=1}^{d} \varphi_{ir} + \sum_{j=1}^{c} y_{kj},$$

the objective function

$$Min \sum_{i=1, i \neq k}^{n} (\Delta_i^O + \Delta_i^I),$$

can be rewritten as

$$Min \sum_{i=1, i \neq k}^{n} \left(\sum_{j=1}^{m} v_j x_{ij} - \sum_{r=1}^{s} u_r y_{ir} + \sum_{r=1}^{d} \varphi_{ir} + \sum_{j=1}^{c} y_{kj} \right).$$

Also by $(M4)$

$$\Gamma_{kj} = \sum_{i=1, i \neq k}^{n} y_{ij},\ \Phi_{kr} = \sum_{i=1, i \neq k}^{n} \varphi_{ir},\ \text{and} \sum_{r=1}^{d} \Phi_{kr} + \sum_{j=1}^{c} \Gamma_{kj} = \sum_{j=1}^{m} v_j x_{kj} - \sum_{r=1}^{s} u_r y_{kr}.$$

Thus, the new objective function is

$$Min \sum_{i=1}^{n} \left(\sum_{j=1}^{m} v_j x_{kj} - \sum_{r=1}^{s} u_r y_{kr} \right). \tag{2.14}$$

Incorporating Eqns. (2.13) and (2.14) into $(M4)$, the model $(M4)$ can be re-written as the following equivalent LP model $(M5)$. The new objective function is defined by equation (2.14), and equation (2.13) gives us the second constraint of the following model $(M5)$.

$$(M5)\quad Min \sum_{i=1}^{n}\left[\sum_{j=1}^{m} v_j x_{ij} - \sum_{r=1}^{s} u_r y_{ir}\right]$$

subject to

$$\sum_{j=1}^{m} v_j x_{kj} - \sum_{r=1}^{s} u_r y_{kr} - \sum_{r=1}^{d} \Phi_{kr} - \sum_{j=1}^{c} \Gamma_{kj} = 0,$$

$$\sum_{j=1}^{m} v_j x_{ij} - \sum_{r=1}^{s} u_r y_{ir} + \sum_{r=1}^{d} \varphi_{ir} + \sum_{j=1}^{c} \gamma_{ij} \geq 0,\ i = 1,\ldots,n;\ i \neq k,$$

$$\sum_{j=1}^{m} v_j x_{kj} = 1,$$

$$\Gamma_{kj} = \sum_{i=1, i \neq k}^{n} \gamma_{ij},$$

$$\Phi_{kr} = \sum_{i=1, i \neq k}^{n} \varphi_{ir},$$

$$\Gamma_{kj} \leq v_j x_{kj} - \varepsilon,\ j = 1,\ldots,c,$$

$$\varphi_{ir} \leq u_r y_{ir} - \varepsilon,\ r = 1,\ldots,d;\ i = 1,\ldots,n;\ i \neq k,$$

$$v_j, u_r, \Gamma_{kj}, \gamma_{ij}, \Phi_{kr}, \varphi_{ir}, \varepsilon \geq 0;\ i = 1,\ldots,n;\ j = 1,\ldots,m;\ r = 1,\ldots,s.$$

Model $(M5)$ is an LP model. By applying this model any k^{th} DMU under common weights can improve its efficiency by reducing inputs under CSOI, or increasing output under CSOO, or both. The model is feasible since it is the linear form of model $(M3)$, which has already been proved to be feasible. However, model $(M5)$ does not minimize the amount of input/output transfer. Let $\theta_i^{CW*}(i = 1,\ldots,n)$ be the common-weight efficiency of all DMUs after the applying model $(M5)$. We now design a model to minimize the input/output transfer while ensuring that the efficiency of the DMUs does not reduce any further. This LP model $(M6)$ is as follows:

$$(M6)\quad Min \sum_{r=1}^{d} \Phi_{kr} + \sum_{j=1}^{c} \Gamma_{kj}$$

subject to

$$\sum_{j=1}^{m} v_j x_{kj} - \sum_{r=1}^{s} u_r y_{kr} - \sum_{r=1}^{d} \Phi_{kr} - \sum_{j=1}^{c} \Gamma_{kj} = 0,$$

$$\theta_i^{CW*}\left[\sum_{j=1}^{m} v_j x_{ij} + \sum_{j=1}^{c} \gamma_{ij}\right] - \sum_{r=1}^{s} u_r y_{ir} + \sum_{r=1}^{d} \varphi_{ir} \leq 0,\ i = 1,\ldots,n;\ i \neq k,$$

$$\sum_{j=1}^{m} v_j x_{ij} + \sum_{j=1}^{c} \gamma_{ij} - \sum_{r=1}^{s} u_r y_{ir} + \sum_{r=1}^{d} \varphi_{ir} \geq 0,\ i = 1,\ldots,n;\ i \neq k$$

$$\sum_{j=1}^{m} v_j x_{kj} = 1,$$

$$\Gamma_{kj} = \sum_{i=1, i \neq k}^{n} \gamma_{ij},$$

$$\Phi_{kr} = \sum_{i=1, i \neq k}^{n} \varphi_{ir},$$

$$\Gamma_{kj} \le v_j x_{kj} - \;, \; j = 1, \ldots, c,$$

$$\varphi_{ir} \le u_r y_{ir} - \;; \; r = 1, \ldots, d; \; i = 1, \ldots, n; \; i = k,$$

$$v_j, u_r, \Gamma_{kj}, \gamma_{ij}, \Phi_{kr}, \varphi_{ir}, \quad \ge 0; \; i = 1, \ldots, n; \; j = 1, \ldots, m; \; r = 1, \ldots, s.$$

Applying model ($M5$), followed by model ($M6$), we can calculate the minimum necessary input/output change that allows the observed DMU k to become e□cient, without reducing overall e□ciency in the system.

3 Numerical Example

The example uses data for 14 hospitals, with the inputs being the number of doctors (Input 1), nurses (Input 2) and the outputs being the number of outpatients (Output 1), inpatients (Output 2). The data is obtained from the work of Cooper et al. (2007). In this example, we assume that the number of doctors (Input 1) and inpatients (Output 2) are under the CSOI and CSOO constraint respectively, and that the DMUs are under Common Weights (CW). These assumptions are made only as part of the demonstration. The data for the DMUs, as well as the original e□ciency score under CW, is shown in the table below.

Table 1: Input, Output, and Common Weight E□ciency Data of 14 Hospitals

and the efficiency of all DMUs is measured using a common set of weights. While there exists work on the common set of weights problem in DEA, and the constant sum of inputs/outputs problem, this paper's models address situation where both constraints are in operation. The problem is solved with a two-step process using two LP models. The first model determines the necessary amount of input/output change in the observed DMU which leads to it achieving efficiency. The second LP model determines the minimum change in each parameter while ensuring that overall efficiency in the system does not suffer. These models address a gap in existing literature, and can be applied in any situation where a single DMU is seeking to improve efficiency when there is a limit on total input/output, and parameter weights have a common value. It may be noted that while the observed DMU's efficiency and the overall efficiency has been improved, some individual DMUs may suffer efficiency reduction. A future direction of research may be to improve the observed DMU and overall efficiency, without reducing efficiency in any other DMU.

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