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Characterization of maxmed mechanisms for multiple objects

Characterization of maxmed mechanisms for multiple objects

Ranojoy Basu [*1] and Conan Mukherjee [”]

in the class of anonymous, non-envious, feasible, individually rational, and strategyproof mechanisms. In the present paper, we extend his results to the multiple identical object setting by providing an extension of the maxmed mechanism functional form to the multiple object setting.

Allowing multiple objects complicates the analysis substantially as any one agent getting an object no longer implies that every other agents gets no object. Furthermore, with multiple objects, there is a proliferation of allocation choices available to the planner at any reported valuation profile, because now she can choose **not** to allot all available objects.[2] Hence, to obtain a characterization on the lines of Sprumont [23] with multiple objects, it becomes necessary to use a restriction on the behaviour of mechanisms as the number of objects being allocated changes. To accommodate such a restriction, we study the problem in terms of "families" of mechanisms which contain a specific mechanism for each possible number of units k that may be available for allocation. Thus, in our setting, a social planner must choose a family of mechanisms to execute the allocation exercise prior to the realization of the actual number of objects to be allocated.[3]

This conceptualization of families of mechanisms allows us to motivate a *regularity* condition, also used in Basu and Mukherjee [4], which requires that set of valuation profiles where no objects are allocated - to not shrink when the number of units available for allocation increases.[4] We analyze the class of regular families, which contain continuous, anonymous, feasible, individually rational, strategyproof mechanisms that satisfy non-bossiness in decision.[5] In particular, we identify families $\mathcal{F}$ which are Pareto optimal among all families that comprise of anonymous, continuous, feasible, individually rational, non-bossy in decision, non-envious, and strategyproof mechanisms. We show that these Pareto optimal families are same as the ones that use maxmed mechanisms to allocate different supplies of available objects while using the *same* non-negative reserve price. We call these the 'maxmed families' of mechanisms, and thus, present a complete characterization of the maxmed families.

Anonymity requires that the welfare obtained from bidding in a mechanism not depend on agent identities. Non-bossiness in decision requires that no agent be able to influence allotment decision of another agent without changing her own allotment decision.[6] Feasibility requires that the mechanism not entail wastage (so that sum of transfers

[2] So if three objects are available, then she can choose to allocate any $k \in \{0,1,2,3\}$ objects.

[3] Such a setting is observed in many real life situations. For example, an auctioneer (government

is never positive); while individual rationality implies that agents are not penalized for participating in the mechanism (so that utility obtained by bidding is never negative). Continuity of a mechanism ensures that mechanism outcomes do not change arbitrarily for small changes in bid values, and strategyproofness ensures truth telling is a weakly dominant strategy for all agents in the ensuing message game.

2 Literature Review

As mentioned above, our work is an extension of Sprumont [23] to the multiple identical object setting. Apart from Sprumont [23], our work also relates to the papers on optimal strategyproof mechanisms to allocate multiple objects. Some such notable papers are Apt, Conitzer, Guo and Markakis [1], Athanasiou [3], Guo and Conitzer [11], Guo and Conitzer [12], Moulin [16], Moulin [17], Ohseto [20]. While these papers di er in terms of the class of mechanisms considered and the optimality notion used, all of them assume allotment decision e ciency and hence, limit their study to the class of VCG mechanisms (Vickrey [27], Clarke [5], Groves [10]).

Some papers which consider the problem of welfare maximization while allowing for deterministic mechanisms without allotment e ciency are: de Clippel, Naroditskiy and Greenwald [6], Drexl and Kleiner [7], Shao and Zhu [22]. Drexl and Kleiner [7] focuses on strategyproof, individually rational and feasible mechanisms in a two agent setting, egyproof mechanisms. Thus, our paper specifies the exact functional form of maxmed mechanisms when extended to the multiple homogeneous object allocation setting.

3 Model

We consider a situation where m homogeneous indivisible objects are to be allotted to agents in $N = \{1, 2, \ldots, n\}$ with unit demand with $m < n$. Each agent $i \in N$ has an independent private valuation $v_i \in \mathbb{R}_+$. For any $i \in N$, a generic allocation of i is denoted by (d_i, t) where d_i represents the object allotment decision taking values in $\{0, 1\}$ with $d_i = 1$ if and only if i gets an object, and t represents an amount of money. We assume that agents have quasilinear preferences over object and money, that is, utility to i from the allocation (d_i, t) is $d_i v_i + t$.

A mechanism is a tuple of functions (d^m, τ^m) such that at any reported profile of valuations $v \in \mathbb{R}^N_+$, each agent i is allocated a monetary transfer $\tau^m_i(v) \in \mathbb{R}$ and a decision $d^m_i(v) \in \{0, 1\}$. For any reported valuation profile $v \in \mathbb{R}^N_+$, define $W^m(v) := \{i \in N \mid d^m_i(v) = 1\}$ to be the set of agents that are allocated an object. Note that at any reported profile of valuations $v \in \mathbb{R}^N_+$, $|W^m(v)| \le m$, that is, all objects need not get allocated at all reported profiles. Therefore, the utility to any agent i with a true valuation of v_i at any reported profile $v' \in \mathbb{R}^N_+$, from the mechanism (d^m, τ^m) is given by $u((d^m_i(v'), \tau^m_i(v')); v_i) = v_i d^m_i(v') + \tau^m_i(v')$. For any $m \ge 2$, let A^m be the set of all possible mechanisms to allocate m objects.

As mentioned earlier, in this paper, we focus on a family of mechanisms that describe procedures to allocate any number of homogeneous objects. Such a family is a list of mechanisms specifying one mechanism for each possible quantity of homogeneous object supply. Thus, a family of mechanisms represents an *ex-ante* sale procedure, that is chosen and fixed prior to the realization of the number of objects to be available for allotment.[8] Let A be the set of all such families of mechanisms, that is, $A := \bigcup_{m \in \mathbb{N}} A^m$. Also, let $F = \{F^1, F^2, \ldots\}$ denote a generic family of mechanisms in A, with the interpretation that the mechanism F^m is to be used to allot objects when the number of available objects turns out to be m. In this paper, we focus on well behaved families of mechanisms which

Let A_r denote the set of regular families of mechanisms.

Thus, a regular family of mechanisms displays a monotonicity property such that the set of

In the second definition below, we state the extension of the class of *maxmed* mechanisms, which were introduced by Sprumont [23] for a single object setting, to the present multiple identical object setting.

Definition 3. Any mechanism $(d^{m,r}, \tau^{m,r}) \in \mathcal{A}^m$ is said to be a *maxmed with reserve price* $r \geq 0$ if for all $i \in N$ and all $v \in \mathbb{R}^N_+$,

- $v_i < \max\{v_{-i}(m), r\} \Rightarrow d_i^{m,r}(v) = 0$
- $v_i > \max\{v_{-i}(m), r\} \Rightarrow d_i^{m,r}(v) = 1$
- $$\tau_i^{m,r}(v) = \begin{cases} \text{med}\left\{0, v_{-i}(m) - r, \frac{mr}{n-m}\right\} & \text{if } d_i^{m,r}(v) = 0 \\ \text{med}\left\{0, v_{-i}(m) - r, \frac{mr}{n-m}\right\} - \max\{v_{-i}(m), r\} & \text{if } d_i^{m,r}(v) = 1. \end{cases}$$

For any non-negative real number r, let $F_{M,r}$ be a family of mechanisms such that for any m, $F^m_{M,r}$ is a maxmed mechanism with reserve price r. Thus, $F_{M,r}$ represents an ex-ante maxmed sale procedure with reserve price r. Let $\mathcal{M} := \{F_{M,r}\}_{r \geq 0}$ be the set of all such maxmed sales procedure.

Now, we define a popular strategic axiom in the independent private values setting, strategyproofness, which eliminates any incentive to misreport valuation for each agent by making it weakly dominant strategy to reveal her true valuation in the ensuing message game.

Definition 4. A mechanism $(d^m, \tau^m) \in \mathcal{A}^m$ satisfies *strategyproofness* (SP) if for all $i \in N$, all $v_i, v_i' \in \mathbb{R}_+$, and all $v_{-i} \in \mathbb{R}^{N \setminus \{i\}}_+$,

$$u(d_i^m(v_i, v_{-i}), \tau_i^m(v_i, v_{-i}); v_i) \geq u(d_i^m(v_i', v_{-i}), \tau_i^m(v_i', v_{-i}); v_i).$$

Next, we define the axiom of 'non-bossiness in decision' which requires (only) the decision rule in a mechanism to be well-behaved in the sense that no agent is able to influence allotment decision of another agent without changing her own allotment decision.

Definition 5. A mechanism $(d^m, \tau^m) \in \mathcal{A}^m$ satisfies *non-bossiness in decision* (NBD) if for all $i \in N$, all $v \in \mathbb{R}^N_+$ and all $v_i' \in \mathbb{R}_+$,

$$d_i^m(v) = d_i^m(v_i', v_{-i}) \Rightarrow d_j^m(v) = d_j^m(v_i', v_{-i}); \forall j \neq i.$$

and second highest bidder whenever either of their bids is greater than or equal to 20, or else no objects are allocated. Further, any agent who is not allocated an object receives zero transfer, while any agent who is allocated an object pays a price equal to: 20 if bids of all other agents are strictly less than 20, or else the third highest bid. To see that this mechanism is discontinuous, consider a sequence of profiles $\{(20 - \frac{1}{k}, 6, 5)\}_k$. Note that for all k, the agent 2 does not get an object, but she gets an object at the limit profile $(20, 6, 5)$. However, 2 is charged a price 5 at the limit, which makes her prefer getting the object to not getting the object, that is, $u_2((1, -5), 6) > u_2((0, 0), 6)$.

As noted in Thomson [25], NBD represents a strategic hindrance to collusive practices where agents form groups to misreport their valuations in a coordinated manner so that object allotment decision of any one member changes to her benefit, while others' remain unchanged.

The following three axioms represent three different notions of fairness. The first of these states the concept of anonymity which requires that utility derived from an allocation by any agent be independent of her identity.[11] The second one presents the fairness notion that each agent should have some opportunity to win an object, irrespective of other agents' reports.[12] Finally, the third axiom states the notion of no-envy which requires that every agent prefers her own allocation (of decision and transfer from the mechanism) to that of any other agent.[13]

Definition 6. A mechanism $(d^m; {}^m)$

Finally, in the axiom below, we present the fairness notion that requires all agents to

Fact 2. Fix any family $F \in A_c \cap A_r$. For any m, if the mechanism F^m satisfies AN, AS, NBD and SP, then there exists an r 0 such that for all $i \in N$ and all $v \in \mathbb{R}^N_+$,

- $T_i^m(v_{-i}) = \max\{v_{-i}(m); r\}$ and
- $K_i^m(v_{-i}) = K^m($

same decision also get the same transfer, and so $\phi_1(v) = \phi_{k+1}(v)$.[17] But this implies that $z_m + K^m(z) = z_m + K^m(x_k^0; z_{-k})$ and hence, we get a contradiction. Now, suppose that (ii) does not hold. That is, there exists a $k \in \{m+1; \ldots; n-1\}$ and an $x_k^{00} < z_m$ such that $K^m(x_k^{00}; z_{-k}$

$k \in \{1, \ldots, t\}$ and an $x_k^0 > z_k$ such that $K^m(x_k^0$

the multiple object version of the maxmed mechanisms introduced by Sprumont [23] for a single object setting. We rst de ne the notion of Pareto dominance in a class of mechanisms. For any given supply of objects m, and any set of mechanisms S^m, de ne a weak partial order on S^m in the following manner. For any two mechanisms $(d^m;\ ^m);(d^{0m};\ ^{0m})\ 2\ S^m$, let $(d^m;\ ^m)\quad (d^{0m};\ ^{0m})$ i for all $i\ 2\ N$ and all $v\ 2\ R^N_+$, $u(d^m_i(v);\ ^m_i(v);v_i)\quad u(d^{0m}_i(v);\ ^{0m}_i(v);v_i)$. If in addition, this inequality is strict for some i and some v, then we write that $(d^m;\ ^m)\quad (d^{0m};\ ^{0m})$ and say that $(d^m;\ ^m)$ Pareto dominates $(d^{0m};\ ^{0m})$. On the other hand, if $u(d^m_i(v);\ ^m_i(v);v_i) = u(d^{0m}_i(v);\ ^{0m}_i(v);v_i)$ for all i and all v, then we write that $(d^m;\ ^m)\quad (d^{0m};\ ^{0m})$ and say that $(d^m;\ ^m)$ is Pareto equivalent to $(d^{0m};\ ^{0m})$. Finally, we call the class of mechanisms in S^m that are not dominated by any other mechanism in S^m, as the set of Pareto optimal mechanisms in S^m.

Now, we de ne our notion of Pareto optimal families of mechanisms. For any given [0.204(as)-369(the)-370(se6)]TJ/F29 11 set of families F, de ne a weak partial order `r ' over F, w9898 Tf 11.664 4.338 Td 69dr(0)]T0

the set $P_z := \{v \in \mathbb{R}^N_+ | \exists\, i \in N$ such that $v_i = z\}$, and for all $v \in P_z$, define the set $a^v_z := \{i \in N | v_i = z\}$. Therefore, by Fact 1, P_z is the set of all possible profiles v such that all agents i in a^v_z are assigned the following transfer by mechanism F^m,

$$ {}^{m}_{i}(v) = \begin{cases} K^m(z) & \text{if } d^m_i(v) = 0 \\ K^m(z) \quad \max\{z(m); r\} & \text{otherwise.} \end{cases} $$

Now, construct another family F'' such that $F''^k = F^k$ for all $k \neq m$, and $F''^m := (d''^m; {}''^m)$ satisfies the following properties:

- $(d''^m_i(v); {}''^m_i(v)) = (d^m_i(v); {}^m_i(v))$ for all $i \in N$ and all $v \in \mathbb{R}^N_+ \setminus P_z$,
- $d''^m_i(v) = d^m_i(v)$ for all $i \in N$ and all $v \in P_z$,
- ${}''^m_i$

above cases. Thus, we can infer that $(d^{m;r}; \; ^{m;r})$ satisfies feasibility, NE and IR. Further, it is easy to see that $(d^{m;r}; \; ^{m;r})$ satisfies NBD, and SP. To see that $(d^{m;r}; \; ^{m;r})$ satisfies continuity, note that the premise of the continuity condition applies only if the limit profile $\tilde{v}$ (of the chosen sequence) is such that there exists an i such that $\tilde{v}_i = \max\{\tilde{v}_{-i}(m); r\}$, in which case $u((1; \; _i(\tilde{v}; d_i = 1); \tilde{v}_i) = u((0; \; _i(\tilde{v}; d_i = 0); \tilde{v}_i) = \text{med}\; 0; v_{-i}(m) \quad r; \frac{mr}{n \; m}$. Finally, it is easy to see that $F_{M;r} \; 2 \; A_r$, because the set of profiles where no object gets allocated is $[0; r)^n$, which remains unchanged as m increases.

To complete the proof of sufficiency, we now need to show that $F_{M;r}$ is Pareto undominated in $\tilde{A}$. To prove this, suppose the contrapositive, that is, suppose that there exists a family of mechanisms $\hat{F} \; 2 \; \tilde{A}$ such that $\hat{F} \; s \; F_{M;r}$. This supposition implies that there exists an m^0 such that $\hat{F}^{m^0} := (\hat{d}^{m^0}; \hat{\ }^{m^0}) \quad F^{m^0}_{M;r} = (d^{m^0;r}; \; ^{m^0;r})$. Now, since $\hat{F} \; 2 \; \tilde{A}$, we can infer from Proposition 2 that there exists an $\hat{r} \quad 0$ such that for all i and all v, the associated threshold function $\hat{T}^{m^0}(v_{-i}) = \max\{v_{-i}(m^0); \hat{r}\}$ and the associated $\hat{K}^{m^0}$ function satisfy the conditions (A), (B), and (C) of Proposition 2. Now, from the proof of necessity we can infer that for any k objects, $F^k_{M;\hat{r}}$ is either Pareto equivalent to $\hat{F}^k$ or else Pareto dominates $\hat{F}^k$. Therefore, we can infer that $F_{M;\hat{r}} \; r \; \hat{F}$, and so, by supposition, $F_{M;\hat{r}} \; s \; F_{M;r}$. This implies that $\hat{r} \neq r$. If $\hat{r} > r$, then fix $m = 2$ and consider a profile v such that $v_1 > \dots > v_n$ and for all i, $v_i \; 2 \quad r; \min \frac{nr}{n \; m}; \hat{r}$. It is easy to see that $u(F^2_{M;r}(v); v_n) = v_m \quad r > u(F^2_{M;\hat{r}}(v); v_n) = 0$, which contradicts $F_{M;\hat{r}} \; s \; F_{M;r}$. Similarly, if $\hat{r} < r$, then again fix $m = 2$ and consider a profile $\tilde{v}$ such that for all i, $\tilde{v}_i = \frac{nr}{n \; m} + 1$. Once again we get that $u(F^2_{M;r}(v); v_n) = \frac{mr}{n \; m} > u(F^2_{M;\hat{r}}(v); v_n) = \frac{m\hat{r}}{n \; m}$, which contradicts $F_{M;\hat{r}} \; s \; F_{M;r}$. Thus, we get a contradiction in both cases, which implies that $F_{M;r}$ is Pareto undominated in $\tilde{A}$, and so, $F_{M;r} \; 2 \; A$. □

Now, it is easy to see that no individually rational mechanism can be Pareto dominated by another mechanism that does not satisfy individual rationality. Hence, we can easily infer that within the class of families $\hat{A} \quad A_c \setminus A_c$ that comprise of mechanisms satisfying AN, feasibility, NBD, NE and SP; the set of maxmed families M is Pareto optimal - but not uniquely Pareto optimal.

5 Conclusion

In this paper, we provide an extension of maxmed mechanisms to the multiple homogeneous objects setting. We conduct our analysis in terms of families of mechanisms which we interpret as ex-ante sale procedures that list a separate mechanism to be used to allocate different possible supplies of the homogeneous objects.

We consider a regular class of families of continuous mechanisms that satisfy anonymity, feasibility, individual rationality, no-envy, non-bossiness in decision and strategyproofness. We show that the maxmed sale procedures, that is, the families which use maxmed

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