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Pareto optimal anonymous mechanisms

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Abstract

This paper presents a new characterization of \maxmed" mechanisms introduced by Sprumont [26]. This paper, in a two agent setting, shows that maxmed mechanisms are the unique Pareto optimal mechanisms among all mechanisms that satisfy anonymity, strategyproofnes, nonbossiness in decision, feasibility and individual rationality.

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1 Introduction

Sprumont [26] studies the important problem of identifying Pareto optimal mechanisms for the single object allotment problem with money. He obtains a remarkable partial result by introducing a new class of \maxmed" mechanisms that are the only Pareto optimal mechanisms in the class of anonymous, non-envious, feasible and individually rational strategyproof mechanisms. In the present paper, I provide a similar, but independent characterization of maxmed mechanisms that does not use the axiom of no-envy.[1] In particular, I consider the class of mechanisms that satisfy anonymity in welfare, feasibility, individual rationality, non-bossiness in decision, and strategyproofness. I identify the unique Pareto optimal mechanisms in this class as the class of maxmed mechanisms. I use a two agent setting that can be applied to practical situations like: bilateral trading over an indivisible object between a buyer and a seller, allotment of a government license to private buyers, bankruptcy auction of capital assets by lenders etc.

Anonymity is a popular fairness axiom that requires allocations from a mechanism to any agent be independent of the social identity of the agent, and depend only on the bid values received by the planner.[2] Strategyproofness is a popular strategic axiom

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[1]No-envy is a well known fairness notions that imposes strong technical restrictions on the decision and the transfer functions of a mechanism. It requires that at any state of nature, no agent strictly

that requires mechanims to induce agents to bid their true valuations in the ensuing object characterization of mechanisms using a reserve price object allocation rule, which was presented in Basu and Mukherjee [4]; and use the additional axioms of feasibility, individual rationality, and Pareto optimality to characterize the maxmed mechanisms. Like Sprumont [26], I show that maxmed mechanisms continue to be Pareto optimal in the class of mechanisms satisfying anonymity, feasibility, non-bossiness in decision and strategyproofness.

2 Model

Consider a 2 agent model with set of agents $N = \{1, 2\}$ and an indivisible object. Each agent i has a private valuation $v_i \geq 0$ for the object. A mechanism is a tuple (d, τ) such that at any reported profile of valuations $v \in \mathbb{R}^N_+$, each agent i is allocated a transfer $\tau_i(v) \in \mathbb{R}$ and a decision $d_i(v) \in \{0, 1\}$ such that $\sum_{i \in N} d_i(v) \leq 1$. I follow the notation where $d_i(v) = 1$ implies that agent i gets the object, while $d_i(v) = 0$ stands for i not getting the object. Note that I assume that the object may remain unallocated at some profile of reported valuations. Define $w(v)$ to be the agent getting the object at any profile v.[7] The utility to agent i with a true valuation of v_i at any reported profile $v' \in \mathbb{R}^N_+$, from a mechanism (d, τ) is given by $u(d_i(v'), \tau_i(v'); v_i) = v_i d_i(v') + \tau_i(v')$. Let $\forall i \neq j \in N$, $\forall v \in \mathbb{R}^N_+$, $v_{-i} = v_j$, and define the median of any three real numbers x, y, z

mechanism allocation decision, irrespective of what other agents are bidding.[8]

Definition 1. A mechanism (d, τ) satisfies *agent sovereignty* (AS) if for all $i \neq j \in N$ and all $v \in \mathbb{R}^N_{++}$, $\exists\, v'_i \geq 0$ such that

$$d_i(v) \neq d_i(v'_i, v_j).$$

As shown in Proposition 1, I get agent sovereignty in this paper for free as it is implies by the other axioms defined below.

I also use the following notion of fairness which requires that utility derived from an allocation by any agent be independent of her identity. Thus, any discrimination across agents in terms of utilities received from the mechanism must only be in terms of their valuations for the object. Any mechanism violating this property is likely to be unacceptable in modern societies built upon the inalienable right to equality.

Definition 2. A mechanism (d, τ) satisfies *anonymity in welfare* (AN) if for all $i \in N$, all $v \in \mathbb{R}^N_+$ and all bijections $\pi : N \mapsto N$,

$$u(d_i(v), \tau_i(v), v_i) = u(d_{\pi i}(\pi v), \tau_{\pi i}(\pi v), v_{\pi i}),$$

where $\pi v := \left(v_{\pi^{-1}(k)}\right)_{k=1}^{n}$.

Now, I define a popular strategic axiom in the independent private values setting, strategyproofness, which eliminates the incentive to misreport valuation for each agent by making it a weakly dominant strategy to reveal her true valuation in the ensuing message game.

Definition 3. A mechanism (d, τ) satisfies *strategyproofness* (SP) if $\forall i \in N$, $\forall v_i, v'_i \in \mathbb{R}_+$, $\forall v_{-i} \in \mathbb{R}^{N \setminus \{i\}}_+$,

$$u(d_i(v_i, v_{-i}), \tau_i(v_i, v_{-i}), v_i) \geq u(d_i(v'_i, v_{-i}), \tau_i(v'_i, v_{-i}), v_i).$$

Next, I define the axiom of `non-bossiness in decision' which requires (only) the decision rule in a mechanism to be well-behaved in the sense that no agent is able to influence allocation decision of another agent without changing her own allocation decision.

Definition 4. A mechanism (d, τ) satisfies *non-bossiness in decision* (NBD) if for all $i \in N$, all $v \in \mathbb{R}^N_+$ and all $v'_i \in \mathbb{R}_+$,

$$d_i(v) = d_i(v'_i, v_{-i}) \implies d_j(v) = d_j(v'_i, v_{-i}), \forall j \neq i.$$

[8]Similar axioms have been used by Marchant and Mishra [16] and Moulin and Shenker [20].

As noted in Thomson [28], NBD represents a strategic hindrance to collusive practices where agents form groups to misreport their valuations in a coordinated manner so that object allocation decision for any one member changes to her benefit, while others are not worse off.

The following axiom of feasibility requires that the sum of transfers not exceed zero for any profile of valuations and thus, ensures that implementing fair mechanisms do not entail wastage of resources.

Definition 5. A mechanism $(d; \tau)$ satisfies *feasibility* if for all $v \in \mathbb{R}^N_+$,

$$\sum_{i \in N} \tau_i(v) \leq 0.$$

In the final axiom below, I present the fairness notion that requires all agents to get a non-negative utility at all possible profiles so that voluntary participation in the mechanism can be ensured.

Definition 6. A mechanism $(d; \tau)$ satisfies *individual rationality* (IR) if for all $i \in N$, all $v \in \mathbb{R}^N_+$,

$$v_i d_i(v) + \tau_i(v) \geq 0.$$

To conceptualize the Pareto frontier of any class of mechanisms S, I define a *weak partial order* $\succsim$ on the mechanisms in S in the following manner. For any two mechanisms $(d; \tau); (d'; \tau') \in S$, let $(d; \tau) \succsim (d'; \tau')$ iff for all $i \in N$ and all $v \in \mathbb{R}^N_+$, $u(d_i(v); \tau_i(v); v_i) \geq u(d'_i(v); \tau'_i(v); v_i)$. If in addition, this inequality is strict for some i and some v, then I write that $(d; \tau) \succ (d'; \tau')$ and say that $(d; \tau)$ *Pareto dominates* $(d'; \tau')$. On the other hand, if $u(d_i(v); \tau_i(v); v_i) = u(d'_i(v); \tau'_i(v); v_i)$ for all i and all v, then I write that $(d; \tau) \sim (d'; \tau')$ and say that $(d; \tau)$ is *Pareto equivalent* to $(d'; \tau')$. Finally, I call the class of mechanisms in S that are not dominated by any other mechanism in S, as the set of *Pareto optimal* mechanisms in S.

3 Results

I begin by presenting a well known result which states that the decision rule associated with a strategyproof mechanism must be non-decreasing in one's own reported valuation.[9] More specifically, $\forall i$ and $\forall v_{-i}$, there exists a finite threshold price $T_i(v_{-i})$ such that: i wins an object if $v_i > T_i(v_{-i})$, and fails to win an object if $v_i < T_i(v_{-i})$.

[9] This result can be found as Proposition 9.27 in Nisan [22] and Lemma 1 in Mukherjee [21].

$i = 1; 2,\ u_i(d_i^0(v);\ {}_i^0(v); v_i) - u_i(d_i(v);\ {}_i(v); v_i$

$r = u_i(d_i(\hat{v}); \ _i(\hat{v}); \hat{v}_i)$, or else $u_h(d'_h(\hat{v}); \ '_h(\hat{v}); \hat{v}_h) = u_h(d_h(\hat{v}); \ _h(\hat{v}); \hat{v}_h); \forall h \in N$. Thus, $(d'; \ ') \quad (d; \ )$, which implies that $(d; \ )$ is not Pareto optimal, and hence, I get a contradiction. Therefore, by (ii), I can infer that (iii) $K(x) = r; \forall x \geq 2r$.

Now consider any $z \in [r; 2r)$, and consider a pro le v with $v_i \geq 2r$ and $v_j = z$. It is easy to see that (iii), feasibility and IR imply that $0 \leq K(z) \leq z - r$. Now, if there exists a $z' \in [r; 2r)$ such that $K(z') \in (0; z' - r)$, I can construct another mechanism $(d''; \ '')$ such that,

- $(d''(v); \ ''(v)) = (d(v); \ (v))$ for all v such that for any $i \neq j$, Finally, to prove that $(d^r; \tau^r)$ is Pareto optimal, suppose the contrapositive - that is, suppose that there exists a $(d; \tau) \in$ such that $(d; \tau) \succ (d^r; \tau^r)$. Then, by the proof of necessity, there exists an $r' \geq 0$ and a maxmed mechanism $(d^{r'}; \tau^{r'}) \in M$ such that $(d^{r'}; \tau^{r'}) \succ (d; \tau)$, and so, $(d^{r'}; \tau^{r'}) \succ (d^r; \tau^r)$. Therefore, I can infer that $r \neq r'$. Now if $r > r'$, then consider a profile v such that $v_i > v_j > 2r$, and note that $u_j(d^r_j(v); \tau^r_j(v); v_j) := r > r' = u_j(d^{r'}_j(v); \tau^{r'}_j(v); v_j)$, which contradicts $(d^{r'}; \tau^{r'}) \succ (d^r; \tau^r)$. Similarly, if $r < r'$, then there exists a profile $\hat{v}$ such that $\hat{v}_i > \max\{r'; 2r\} \geq \min\{r'; 2r\} > \hat{v}_j > r$, and note that $u_i(d^{r'}_i(\hat{v}); \tau^{r'}_i(\hat{v}); \hat{v}_i) = \hat{v}_i - \hat{v}_j + \hat{v}_j - r = \hat{v}_i - r > \hat{v}_i - r' = u_i(d^r_i(\hat{v}); \tau^r_i(\hat{v}); \hat{v}_i)$, which again

5 Conclusion

I present a new characterization of maxmed mechanisms that of Sprumont [26], by substituting the axiom of no-envy with axiom of non-bossiness in decision. I show that in a simple two agent setting, maxmed mechanisms are the only Pareto optimal mechanisms in the class of anonymous, feasible, individually rational, non-bossy in decision, and strategyproof mechanisms. Extension of this characterization to the general n agent setting, or to the multiple object setting is a di cult exercise. I leave these questions for future research.

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