Strategyproof multidimensional mechanism design without unit demand

STRATEGYPROOF MULTIDIMENSIONAL MECHANISM DESIGN WITHOUT UNIT DEMAND

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Abstract. We consider the heterogeneous object auction problem where buyers have private additive valuations and non-unit demand. We completely characterize the class of strategyproof and agent sovereign mechanisms. Further, we introduce a notion of continuity, and show that every continuous, agent sovereign, anonymous and strategyproof mechanism must be e cient. We nd that the only mechanism satisfying these properties is equivalent to operating simultaneous second price auctions for each object - as was done by New Zealand government in allocating license rights to use of radio spectrum in 1990. Finally, we present a complete characterization of simultaneous second price auctions with object speci c reserve prices, in terms of these properties and a weaknon-bossinessrestriction.

JEL classi cation : D44; D47; D63; D71; D82

Keywords: Mechanism design, Heterogeneous objects auction, Non-unit demand, Strategyproofness, Pivotal mechanism.

1. Introduction

We consider the problem where a planner wishes to sell *m* heterogeneous indivisible objects to *n* buyers who have private additive valuations. The planner is unaware of buyers' valuation, and each buyer has linear preferences over objects and money. This problem encompasses many real life applications ranging from spectrum auction to airport landing rights allocation. All such exercises invariably require reporting of personal valuations by interested buyers. How to execute such a sale in a manner that all buyer report truthfully, is an important problem in mechanism design.

In this paper, we adopt the most robust notion of truthful reporting, strategyproofness, which requires that all participants nd it optimal to report their true valuations, irrespective of what all other participants choose to do. As is well known, the remarkable advantage of using this notion of truthful revelation of valuations is that no prior distributional assumptions are required to justify policies implementing strategyproof mechanisms. More speci cally we address the question: what ar26(justify)-equi-343I17.73Si8@iui-3431mequi-c m48 -17427(m483forwhere)-343(3 Td [(@i

Finally, we use an additional axiom, non-bossiness in decision, to completely characterize the class of continuous, agent sovereign, anonymous and non-bossy mechanisms that allow the possibility of some objects remaining unsold. We nd that any such mechanism is equivalent to operating simultaneous sealed bid second price auctions for each available object with varying reserve prices across objects. This non-bossiness axiom is a modi ed version of the conventional non-bossiness axiom of Satterthwaite and Sonnenschein [27], and requires that no buyer be able to a ect the allocation decision of another buyer without a ecting her own allocation decision. As noted by Thomson [30], this axiom, when coupled with strategyproofness, embodies strategic restrictions that discourage collusive practices.

The paper is organized as follows. The next section 2 contains the literature review, section 3 contains model description and relevant de nitions. Section 4 contains the results, section 5 presents a discussion of our results, while section 6 presents the conclusion. Finally, section 7 is the Appendix where proofs and independence of axioms is presented.

2. Relation to literature

Note that our paper assumes that buyers have private additive valuations for the heterogeneous objects up for sale. Some notable papers that analyze sale of heterogeneous objects in the private value setting are: Ausubel [2], Ausubel, Cramton, Pycia, Rostek and Weretka [3], Ausubel and Milgrom [4], Demange, Gale and Sotomayer [9], de Vries, Schummer and Vohra [7], Gul and Stacchetti [12], Mishra and Parkes [19], and Kazumura, Mishra, and Serizawa [14].

Ausubel, Cramton, Pycia, Rostek and Weretka [3] compare rst price and second price auction in terms of the degree of ine ciency in the resultant Bayes Nash equilibrium, and show that expected revenue rankings are ambiguous. Ausubel [2] provides a new interpretation of Walrasian equilibrium by describing a dynamic auction procedure where strategic bidders reveal their preferences over a single price path, and the corresponding Walrasian equilibrium outcome is achieved. Ausubel and Milgrom [4] present a collection of ascending combinatorial bidding auctions where valuations are additive and truthful reporting is a Nash equilibrium.

Demange, Gale and Sotomayer [9] studies a setting where bidders have unit demand, and describes tn adences over come price equilibrium, and show that ascending auctions can implement e cient strategy-proof outcomes only in a unit demand setting.

de Vries, Schummer and Vohra [7] further generalize this setting to unrestricted valuations, and constructs an ascending auction which leads to VCG (Vickrey [31], Clarke [5], Groves [11]) outcome prices when valuations satisfy a `submodularity' property (that is a weaker restriction on valuations than gross substitutes). Mishra and Parkes [19] relax the de Vries, Schummer and Vohra [7] de nition of ascending price auctions suitably, to construct ascending price auctions which maintain single path but attain VCG outcomes for a larger class of valuation functions in ex-post Nash equilibrium.

Unlike all these papers, the main objective of this paper is not to construct or compare auction algorithms. Instead, this paper primarily focusses on the standard direct mechanism design question: what are the strategyproof mechanisms for auctioning heterogeneous objects when buyers have additive valuations and non-unit demand?

To our knowledge, the only paper that considers a similar question is Kazumura, Mishra and Serizawa [14] (henceforth, referred to as [KMS]). They consider mechanisms, which always allot all available objects in a heterogeneous object setting with buyers having unit demand and likes her allocation at least as much as another buyer, and show that such allocations must also be Pareto efficient. Papai [24] looks at strategyproof mechanisms that generate envy-free allocations. She first notes an impossibility where valuation functions are unrestricted, and then identifies a subset of VCG mechanisms that generate envy-free allocations when valuations are superadditive.

Another strand of literature that links to our work is the analysis of monopoly pricing with a single buyer with multidimensional private information. Two notable papers that analyze the problem of expected revenue maximization in a setting where there are several heterogeneous objects need to be sold to a single buyer with additive valuations are: Manelli and Vincent [16] and Rochet and Chone [26]. Both papers address this question under specified distributional assumptions on the multidimensional private information. Our paper, however, adopts a direct mechanism approach which focusses on eliciting true valuation at all states of nature.

We are unaware of any other paper that analyzes heterogeneous object sales from a mechanism design perspective in a private additive valuation setting with multiple buyers and non-unit demand.

3. Model

Fix any $m \geq 2$ and $n \geq 1$. Consider indivisible objects in $M = \{1, 2, \ldots, m\}$ to be sold to buyers in $N = \{1, \ldots, n\}$, where each buyer i has a positive private valuation $v_i^k \in \mathbb{R}_{++}$ for each object $k \in M$. Let $v_i := (v_i^k)_{k \in M}$ denote a typical valuation vector of any buyer i. Let $V_i := \mathbb{R}^m_{++}$ be the set of all such valuation vectors, and let $V := \prod_{i \in N} V_i$ be the set of all possible valuation profiles, where each profile is a $n \times m$ matrix [illegible]

For direct mechanism , let d (v) and p (v) denote the allotment decision matrix and price vector, respectively; corresponding to any valuation pro le v 2 V. Let d_i (v) and p_i (v) be the decision vector assigned to i and the price charged to i, respectively, by mechanism . Further, axiom *agent sovereignty* is a fairness axiom, which describes the individual right of each buyer to target and win any particular object by reporting large enough valuation for it, irrespective of what other buyers are reporting. Any violation of this axiom may lead to situations where some buyer's allotment decision for an object is independent of her valuation for that object.[6]

Definition 1. A mechanism φ satisfies *strategyproofness* (SP) if $\forall i \in N$, $\forall v, v' \in V$ such that $v_{-i} = v'_{-i}$,

$$u(\varphi_i(v); v_i) \geq u(\varphi_i(v'); v_i).$$

Definition 2. A mechanism φ satisfies *agent sovereignty* (AS) if $\forall i \in N$, $\forall k \in M$, $\forall v_i^{-k} \in \mathbb{R}^{m-1}_{++}$, and $\forall v_{-i} \in \prod_{j \neq i} V_j$, there exist $v_i^k, w_i^k > 0$ such that:

$$d_i^k((v_i^k, v_i^{-k}), v_{-i}) \neq d_i^k((w_i^k, v_i^{-k}), v_{-i}).$$

Let $\Phi_{M,N}$ be the set of mechanisms that satisfy SP and AS.

We now define an additional fairness axiom of *anonymity* that is crucially important for public decision making procedures. It requires that utility derived from an allocation by any buyer be independent of her identity. Any mechanism violating this property is highly unlikely to be acceptable in a democratic society.

Definition 3. A mechanism φ satisfies *anonymity in welfare* (AN) if $\forall i \in N$, $\forall v \in V$ and all bijections $\sigma : N \mapsto N$,

$$u(\varphi_i(v); v_i) = u(\varphi_{\sigma(i)}(v^{\sigma}); (v^{\sigma})_{\sigma(i)})$$

where $v^{\sigma} := \left(v_{\sigma^{-1}(t)}\right)_{t=1}^{n}$.

4. Results

4.1. Single buyer case. This section presents results for the simplest case where there is only one buyer. Therefore, the set of all possible decisions $D = \{0, 1\}^m$ and $V = \mathbb{R}^m_{++}$. It is easy to

(1) $d^k(v) = \begin{cases} 1 & \text{if } v^k > T^k \\ 0 & \text{if } v^k < T^k \end{cases}$, and

(2) $p(v) = \sum_{k:d^k(v)=1} T^k$.

Proof of Necessity: Consider any two profiles $v_x; v_y \in V$, and fix any object $k \in M$. SP implies that $u(d(v_x); p(v_x); v_x) \geq u(d(v_y); p(v_y); v_x)$ and $u(d(v_x); p(v_x); v_y)$ (v

compatibility. Further, if $d(v') \neq d(v)$, then,

$$p(v') - p(v) = \sum_{k:d^k(v')=1} T^k - \sum_{k:d^k(v)=1} T^k = \sum_{k:d^k(v')-d^k(v)=1} T^k - \sum_{k:d^k(v')-d^k(v)=-1} T^k$$

Note that for any $k \in M$, if $d^k(v') - d^k(v) = 1$, then $v^k \leq T^k \leq v'^k$, and if $d^k(v') - d^k(v) = -1$, then $v'^k \leq T^k \leq v^k$ (by construction). So,

$$u((d(v'); p(v')); v) - u((d(v); p(v)); v) = \sum_{k:d^k(v')-d^k(v)=1} (v^k - T^k) - \sum_{k:d^k(v')-d^k(v)=-1} (v^k - T^k) \leq 0;$$

and hence, again, there can be no violation of incentive compatibility. Thus, the result follows.

4.2. Multiple buyer case. With multiple buyers, private information in our model becomes an $n \times m$ matrix. We show below how the Theorem 1 can be extended to this general setting.

Theorem 2. A mechanism $\in \Gamma_{M;N}$ if and only if for any $i \in N$, there exist functions $\{T_i^k : \prod_{j \neq i} V_j \to \mathbb{R}_{++}\}_{k \in M}$ such that for all $v \in V$ and all $k \in M$:

(1) $d_i^k(v) = \begin{cases} 1 & \text{if } v_i^k > T_i^k(v_{-i}) \ldots \end{cases}$

1 Tf 6.634 3.056 Td [Tf 5.288 -1.636 Td [( )]4 2.353 0 Td [(d)]TJ86 10.9091 Tf 13.333 0 Td [(v)]TJ/

get that

$$T_i^{k;d_i^{-k}}(v_{-i}) \geq T_i^k(v_{-i})$$

Further, by AS, $T_i^k(v_{-i}) \in \mathbb{R}_{++}$. Thus, arguing as in case (ii) of Theorem 1, the result follows.

Proof of sufficiency: Consider any mechanism $(d;p)$ as described in the statement of the Theorem 2. Fix any buyer i, and any v_{-i}. Since $\{T_i^k(v_{-i})\}_{k \in M}$ values are positive reals, and so, for any k, there exists $\underline{v}_i^k < T_i^k(v_{-i}) < v_i^k$ such that $d((\underline{v}_i^k; v_i^{-k}); v_{-i}) \neq d((v_i^k; v_i^{-k}); v_{-i})$. Thus $(d;p)$ satisfies AS.

Now, consider a valuation profiles $v_i; v_i^0 \in V_i$. Suppose, that v_i is i's true profile, while v_i^0 is a misreport. If $d_i(v_i; v_{-i}) = d_i(v_i^0; v_{-i})$, then by construction, $p_i(v_i; v_{-i}) = p_i(v_i^0; v_{-i})$, and so, there can be no violation of SP. If $d_i(v_i; v_{-i}) \neq d_i(v_i^0; v_{-i})$, then by definition,

$$\begin{aligned} p_i(v_i^0; v_{-i}) - p_i(v_i; v_{-i}) &= \sum_{k: d_i^k(v_i^0; v_{-i})=1} T_i^k(v_{-i}) - \sum_{k: d_i^k(v_i; v_{-i})=1} T_i^k(v_{-i}) \\ &= \sum_{k: d_i^k(v_i^0; v_{-i}) - d_i^k(v_i; v_{-i})=1} T_i^k(v_{-i}) - \sum_{k: d_i^k(v_i^0; v_{-i}) - d_i^k(v_i; v_{-i})=-1} T_i^k(v_{-i}) \end{aligned}$$

between the two as

$$||A - B|| := \sqrt{\sum_{1 \le t \le p} \sum_{1 \le l \le q} (a_{tl} - b_{tl})^2}.$$

Further, a sequence of such matrices A^j is defined to converge to limit matrix A if and only if $(||A^j - A||)_j \to A$.

Definition 4. A mechanism ϕ is said to be continuous if for any $\alpha \in \{0, 1\}$, any $i \in N$, any $k \in M$, and any sequence of profiles $\{v^l\}$ that converges to $\bar{v}$, whenever $d_i^k(v^l) = \alpha$ for all l,

$$d_i^k(\bar{v}) \neq \alpha \implies \Big[u(d_i(\bar{v}); p_i(\bar{v}); \bar{v}_i) = u((\alpha; d_i^{-k}(\bar{v})); p_i(\bar{v}); \bar{v}_i)$$

Remark 2. Note that using a simple m[redacted]to sell m objects is equivalent to using m separate single object auctions a[redacted]ers, one for each di erent object. Further, in the auction for any object k 2[redacted]a positive price $T_i^k(:)$ if and only if she wins k.

Theorem 3. If a mechanism 2 $_{M;N}$, [redacted]

Proof: Fix any mechanism 2 $_{M;N}$, a[redacted] any object k 2 M . The proof is accomplished using the following two st[redacted]

Step 1: In this step, we show tha[redacted]s function.

Suppose not. Then there exis[redacted] v $_{i}$ such that $(T_i^k(v^n{}_i))_n$ does not converge to $T_i(v_i)$. Therefore[redacted]0, and a subsequence $(T_i^k(v^{n_l}{}_i))_l$ such that for all l 2 N, $jT_i^k(v^{n_l}{}_i)$ T[redacted] loss of generality, we assume that for all l 1, (a) $T_i^k(v^{n_l}{}_i)$ $T_i^k($[redacted]e a v_i such that $v_i^t = T_i^t(v_i) + \frac{"}{2}$ for all t 2 M , and consider the s[redacted]$(v_i; v^{n_l}{}_i)$ $_l$. By supposition $(v_i; v^{n_l}{}_i)$ $_l$! $(v_i; v_i)$, and by (a) $_{ij}$ ". [redacted]rality, we assume that for all such

a.0 Tf 9.1ueh2e1(.63-60 [(T 7.585 9.601.15 10.90[redacted]466equence)-328(()]TJ/F36 10.9091 Tf 211.122 0 Td
177 466.675 cm []0 d 0 J 0.436 w55
91 Tf 52.37 0 Td14635 3.055 T2[redacted].381 14137 Td [())]TJ/F38 10.9091 Tf 4.242 0 Td [(j)-278( )
288 -1.636 Td [(7.7]TJ 7.7]090910F37 7.41[redacted][(n)]6 cm 8 5.9776 Tf 5.13F39 7.9701 Tf 5.47 0 Td [(v)]430

Now, fix any $j \neq i$, and define a function $G_j^k(v) : V \mapsto \mathbb{R}$ such that $\forall v \in V$, $G_j^k(v) := v_j^k - T_j^k(v_{-j})$. Note that by (I) and AS,[11]

$$\text{(b)} \quad G_j^k(v) = \begin{cases} \text{negative} & \text{if } v_i^k > T_i^k(v_{-i}) \\ \text{ambiguous} & \text{if } v_i^k \leq T_i^k(v_{-i}). \end{cases}$$

Thus, for any profile v, by construction, $G^k(v)$ cannot depend on v_j^{-k}, while by (b), it depends on v_{-i}. Therefore, condition (b) can hold true only if $T_i^k(v_{-i})$ does not depend on v_j^{-k}. And since i, j, k, and v were chosen arbitrarily, we can infer that $T_i^k(v_{-i}) = f_i(v_1^k, \ldots, v_{i-1}^k, v_{i+1}^k, \ldots, v_n^k)$; $\forall i, k, v$. Hence, the result follows.

Theorem 3 shows that all continuous mechanisms in ${}_{M;N}$ can be implemented via suitably chosen m separate single obEt; 8ob

We now proceed to present the first main result of this paper, which states that the basic ethical notion of anonymity, coupled with the continuity restriction, generates decision efficiency for strategyproof and agent sovereign mechanisms that sell all objects at all profiles. This idea of efficiency is formally defined below.

Definition 6. A mechanism ^{e} is efficient (EFF) if for all $v \in V$,

$$\sum_{i \in N} d_i^{e} v_i = \max_{\hat{d} \in D} \sum_{i \in N} \hat{d}_i v$$

Therefore, for all i and all v,

$$u(\ _i^P(v); v_i) = \sum_{\substack{k \in M \\ d_i^k(v)=1}} \left(v_i^k - \max_{j \neq i} v_j^k \right);$$

that is, the Pivotal mechanism is equivalent to executing m different "Second Price Auction"s - one for each object to be sold.

Note that for any bijection $\ : N \mapsto N$ and any $i \in N; k \in M$; $v_i = (\ v)_{\ i}$, $\max_{j \neq i} v_j^k = \max_{j \neq i} (\ v)_j^k$, and so, $u(\ _i^P(v); v_i) = u(\ _i^P(\ v); (\ v)_{\ i})$. Therefore, it is easy to see that $\ ^P$ satisfies anonymity.

Further, fix any $i \in N$, any $k \in M$, and (without loss of generality) consider any sequence $({}^l v)_l \to \bar{v}$ such that $d_i^{k\ P}({}^l v) = 1$ with $d_i^{k\ P}(\bar{v}) = 0$. Therefore, $(\max_{j \neq i} {}^l v_j^k)_l$ converges to $\max_{j \neq i} \bar{v}_j^k$, and so,

$$d_i^{k\ P}({}^l v) = 1 \text{ for all } l \Rightarrow {}^l v_i^k \geq \max_{j \neq i} {}^l v_j^k \text{ for all } l \Rightarrow \bar{v}_i^k \geq \max_{j \neq i} \bar{v}_j^k.$$

Therefore, $d_i^{k\ P}(\bar{v}) = 0$, which implies $\bar{v}_i^k = \max_{j \neq i} \bar{v}_j^k$, which in turn implies that $u(d_i^{\ P}(\bar{v}); p_i^{\ P}(\bar{v}); \bar{v}_i) = u((1; d_i^{\ k\ P}(\bar{v})); p_i^{\ P}(\bar{v}); \bar{v}_i)$. Thus, $\ ^p$ satisfies continuity. Finally, it is easy to see that Pivotal mechanism satisfies strategyproofness and agent sovereignty. Thus, $\ ^P \in \ _{M;N}$.

Theorem 4 establishes that any anonymous mechanism in $\ _{M;N}$ must be efficient. Therefore, as argued earlier, from Holmström [13] it follows that the only mechanism in $\ _{M;N}$ that is anonymous in our setting, is the Pivotal mechanism. This idea is formalized in the corollary below.

Corollary 2. If a mechanism $\ \in \ _{M;N}$ that sells all objects at all profiles satisfies AN, then $\ = \ ^P$.

Proof: Since buyers pay only if they win an object in our setting, by Holmström [13], Theorem 4 implies that the only anonymous mechanism in $\ _{M;N}$ is the Pivotal mechanism.

Remark 3. As noted in proof of Theorem 4, there may be several different ways of executing such a Pivotal mechanism, each with a separate algorithm to generate an efficient object allocation. One simple and elegant way of implementing Pivotal mechanism is to conduct a separate simultaneous sealed bid second price auction for each object. As noted in Mueller [23], Government of New Zealand used this method to sell cellular management right tenders in 1990.

They were advised this manner of spectrum allocation by the reputed British-American consultancy firm National Economic Research Associates (NERA). Our paper, therefore, provides an axiomatic foundation to this procedural advice.[14]

Now, Theorem 4 focusses on mechanisms that allot all objects at all profiles like KMS [14]. Yet, one could think of mechanisms that, a priori, allow a subset of objects to remain unsold. The most common of such mechanisms would be the reserve price mechanisms, where objects are not sold unless bids received are high enough. The next theorem characterizes these mechanisms using the following non-bossiness property that requires decision functions to be reasonably well behaved, while imposing no restrictions on the transfer function.[16]

Definition 8. A mechanism $= (d;)$ is non-bossy in decision if for any $i \in N$, and any $v, v' \in V$ such that $v_i \neq v'_i$ and $v_{-i} = v'_{-i}$;

$$d_i(v) = d_i(v') \Rightarrow \forall j \neq i; d_j(v) = d_j(v')$$

k is sold to the highest bidder who bids a value for k that is at least as great as r^k, or else it remains unsold,

the winner of k

Case (I): By (b), $d^k(v^0) = 0^n$ for any profile v^0 with $v^{0k} := v^k_1 1^n$ and $v^0_{-k} = v_{-k}$.

Construct a sequence of profiles $\{{}^t w\}_{t=1}^n$ such that ${}^1 w := v^0$, and for all $2 \le t \le n$,

$$ {}^{t-1}w^{-k} = {}^t w^{-k};\ {}^t w^k_t = v^k_t;\ \text{and } {}^{t-1}w^k_{-t} = {}^t w^k_{-t}. $$

By Theorem 2, $d^k_t({}^t w) = 0$ for all $t > 1$, and so, by non-bossiness of decision, Theorem 3 and Theorem 2, as before, any two consecutive profiles in the sequence have the same associated decision matrix. Hence, $d^k({}^t v) = d^k({}^{t+1} v) = 0^n$ for all $t \le n-1$. Since $d^k({}^1 w) = d^k(v^0) = 0^n$, we can infer that $d^k({}^n w) = 0^n$. By construction ${}^n w = v$, and so, we get that (A) $d^k(v) = 0^n$.

Case (II): Consider the profile $\hat{v}$ such that $\hat{v}^{-k} = v^{-k}$ and $\hat{v}^k = \theta 1^n$, where $t^k = v$

Again as argued earlier, by non-bossiness of decision, Theorem 3 and Theorem 2, $d(^{t}v) = d(^{t+1}v)$ for all t n 1, and so, $d_1^k(^1v) = 1 \Rightarrow d_1^k(^nv) = 1$. By construction, $^nv = v$, and so, we get that (C) $d_1^k(v) = 1$.

Recall that the pro e v was chosen arbitrarily without any loss of generality. Therefore, ndings (A) , (B) and (C) taken together imply that; for any object k, there exists a real number r^k 0 such that

$$T_i^k(v_{\ i}) = \max \{ r^k ; \max_{j \neq i} v_j^k \}; \forall i \in N; \forall v \in V:$$

Thus, by Theorem 2, the result follows.

Remark 4. As noted in Remark 3, a simple and elegant manner of implementing mechanisms characterized by Theorem 5 is to hold simultaneous second price auctions with (possibly di erent) reserve prices for each object.

5. Discussion

A setting of heterogeneous object allocation allows us to motivate the notions of `comple-mentarity for non-target objects. Such a behaviour is observed widely enough to be known as `parking' (noted in the Ministry of Business, Innovation & Employment, Government of New Zealand report [25]).[21]

6. Conclusion

In our model of heterogeneous object allocation, we present a characterization of the class of strategyproof and agent sovereign mechanisms. We show that equity and e ciency are closely related, as any anonymous, agent sovereign, continuous and strategyproof mechanism selling all objects must be a decision e cient one. Consequently, by Holmström [13], the only such mechanism in our setting is the Pivotal mechanism. One obvious method of implementing Pivotal mechanism is to conduct separate simultaneous sealed bid second price auctions, as was done by New Zealand government in allocating cellular management rights tenders in 1990. Thus, our results provide an axiomatic justi cation to this method of allocating heterogeneous objects.

We also consider mechanisms that do not sell all objects at all pro les. We show that any such mechanism satis es the aforementioned properties and a non-bossiness property, if and only if it employs object speci c reserve prices, and sells each object to the highest bidder for that object who bids no less than the respective reserve price.

7. Appendix

7.1. Proof of Necessity of Theorem 4. To establish this result, we need to prove the following propositions. Recall that, for any v 2 V, and any i 2 N, $O_i(v)$:= f k 2 M j$d_i^k(v) = 1$ g is the set of objects sold to buyer i at pro le v.

Proposition 1. If a mechanism 2 $_{M;N}$ satis es AN, then for any $x >$ 0, k 2 M and v 2 V such that $v^k = x1^n$;

$$k \; 2 \; O_i(v) \; =) \quad T_i^k(v_{\;\, i}) = x:$$

Proof: Fix any positive real number x. Recall that, by Theorem 3, 2 $_{M;N}$ =) 2 $^S_{M;N}$, and so, $T_i^k(v_{\;\, i}) = f_i(v^k_{\;\, i})$ for any v 2 V, any i 2 N, and any k 2 M. We use this result to accomplish the proof for the following two cases.

Case 1: $m < n$.

Consider a pro le v such that $v_t^l = x$ for all t 2 N and all l 2 M. Now, as $m < n$, there exists a j^0 2 N, such that $O_{j^0}(v)$ = ; , and so, $u(\;_{j^0}(v); v_{j^0}) = 0$. Since satis es AN, $u(\;_{j^0}(v); v_{j^0})$ =

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[21] See footnote 7 in page 8 in Government of New Zealand report [25].

$u(\ _t(v); v_t) = 0$ for all $t \in N$. Therefore, by Theorem 2, it follows that for any buyer t, and any object l: $d^l_t(v) = 1 \implies x = T^l_t(v\ _t)$. By Theorem 3, the result follows.

Case 2: $m \quad n$

Consider the same profile v such that for $v^l_t = x; \forall t \in N; \forall l \in M$. Note that, if there exists any buyer j such that $u(\ _j(v); v_j) = 0$; then, as argued in the previous case, the result follows.

Now, suppose that there does not exist any buyer j with $u(\ _j(v); v_j) = 0$, that is, $u(\ _j(v); v_j) > 0$ for all j.[22] Then, for each $k \in M$; if k is sold at profile matrix v, there exists an $a_{k;v} \in N$ such that $d^k_{a_{k;v}}(v) = 1$, and $\cup_{k \in M} \{a_{k;v}\} = N$. Further, $v^k_{a_{k;v}} = x \quad T^k_{a_{k;v}}(v\ _{a_{k;v}})$ for each sold object k, and $O_t(v) \neq \emptyset$ for each buyer t. Now, fix any $i \in N$ and any $k \in O_i(v)$ such that $x > T^k_i(v\ _i)$.[23]

3, $T_i^k(v_{-i}) = f_i(z)$ and $T_j^k(v_{-i}) = f_j(z)$. Therefore, for economy of notation, henceforth in this subsection, we denote $T_i^k(v_{-i})$ and $T_j^k(v_{-i})$ as $T_i^k(z)$ and $T_j^k(z)$, respectively.

Now, fix any $\in T_i^k(z); T_j^k(z)$, and consider the profile $\hat{v}$ such that for all bui and so, the result follows.

7.2. Independence of axioms.

7.2.1. Theorem 5. We use five axioms in characterizing this result: AS, AN, continuity, nonbossiness and SP. To establish independence between these axioms, we present below five mechanisms, each of which satisfy only four out of the five aforementioned properties.

AN: Say $m = 2$ and $n = 2$. Consider a mechanism where for any k, and any v,

$$T_1^k(v_2) = v_2^k + 1\,;\, T_2^k(v_1) = \max\{0; v_1^k - 1\}:$$

By Proposition 2, this mechanism does not satisfy AN. However, by Corollary 1 and Theorem 2, this mechanism is continuous and strategyproof, respectively. Further, it is easy to see that it satisfies AS and nonbossiness.

AS: Say $m = 2$ and $n = 2$. Consider a mechanism which does not sell any object to any buyer. It is easy to see that this mechanism trivially satisfies AN, nonbossiness, continuity, and SP, but does not satisfy AS.

Continuity: Say $m = 2$ and $n = 2$. Consider a mechanism where for any $i \neq j$, k, and v,

$$T_i^k(v_{-i}) = \begin{cases} 10^k & \text{if } v_j^k \in (0; 10) \\ v_j^k & \text{otherwise.} \end{cases}$$

It is easy to see that this mechanism satisfies AS, AN, SP, and nonbossiness. However, the threshold functions are not continuous, and hence, by Corollary 1, the mechanism does not satisfy continuity.

Nonbossiness: Say $m = 2$ and $n = 3$. Consider a mechanism where for any $i \neq j$, k, and v;

$$T_i^k(v_{-i}) = \begin{cases} \max_{j \neq i} v_j^k + 5 & \text{if } \max_{j \neq i} v_j^k \geq 5 \\ \max_{j \neq i} v_j^k & \text{otherwise} \end{cases}$$

It is easy to see that this mechanism satisfies AN, AS, continuity and SP. However,

$$d^1\begin{pmatrix} 7 & 45 & 30 \\ 6 & 25 & 20 \\ 2 & 15 & 10 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = d^1\begin{pmatrix} 7 & 45 & 30 \\ 4 & 25 & 20 \\ 2 & 15 & 10 \end{pmatrix};$$

and hence, the mechanism violates nonbossiness in decision.

SP: Say $m = 2$ and $n = 3$. Consider a mechanism that sells each object to a highest bidder i for the object at a price that is equal to the amount bid by i. It can easily be seen that this mechanism satisfies AN, AS, continuity and nonbossiness; but does not satisfy SP (as it does not belong to the class characterized by Theorem 2).

7.2.2. Theorem 4. We use five axioms in characterizing this result: AS, AN, continuity, no-wastage (that is, where all objects are sold at all profiles) and SP. As above, to establish independence between these axioms, we present below four example mechanisms which satisfy only four out of the five aforementioned properties. These four examples establish that neither of AS, AN, SP and no-wastage, can be obtained as an implication of the other four axioms.

AN: Say $m = 2$ and $n = 2$. Consider a mechanism where for any k, and any v,

$$T_1^k(v_2) = v_2^k + 1\,;\; T_2^k(v_1) = \max\{0; v_1^k - 1\}:$$

By Proposition 2, this mechanism does not satisfy AN. However, by Corollary 1 and Theorem 2, this mechanism is continuous and strategyproof, respectively. Further, it is easy to see that it satisfies AS and no-wastage.

AS: Say $m = 2$ and $n = 2$. Consider a mechanism which never sells any object to any buyer. It is easy to see that this mechanism trivially satisfies AN, no-wastage, continuity, and SP, but does not satisfy AS.

No-wastage: Say $m = 2$ and $n = 2$. Consider the mechanism such that for any $i \neq j$, k, v;

$$T_i^k(v_{-i}) = \max\{5; v_j\}:$$

It is easy to see that this mechanism satisfies AN, AS, continuity and SP, but does not satisfy no-wastage.

SP: Say $m = 2$ and $n = 3$. As above, consider a mechanism that sells each object to a highest bidder i for the object at a price that is equal to the amount bid by i. It can easily be seen that this mechanism satisfies AN, AS, continuity and no-wastage; but does not satisfy SP (as it does not belong to the class characterized by Theorem 2).

We are unable to present an example to rule out the possibility that any mechanism satisfying

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