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Abstract

We formalize a new concept of `*extended efficiency*' that models important practical cooperative situations which the conventional notion of efficiency fails to accommodate. We use it to completely characterize modifications of Shapley value that satisfy monotonicity and symmetry.

Keywords: Shapley value, extended efficiency, coalitional monotonicity, marginal monotonicity, symmetry

JEL Classification: C71, D60

Introduction

As argued by the seminal work Shapley [1953], application of cooperative game theory to practical situations requires that players be able to evaluate the very "*prospect of having to play a game*". In this paper, we provide a new notion of value using an *extended* notion of efficiency along with monotonicity and symmetry axioms. This extended notion of efficiency requires the sum of individual values to exhaust, not just the grand coalitional worth, but the *sum of worths of all coalitions in a cooperative game*.

This notion of efficiency has received very little attention in the cooperative game theory literature over the years. However, it is quite intuitive and applies to several practical settings.[1] A typical example of such a setting would be a firm whose `line workers' or `partners'

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[1]One may think of modelling these practical settings by accounting a coalition's worth to be sum of worths of its sub-groups. However, as we argue in the Discussion section, such modelling would lead values that are socially unacceptable.

produce multiple products or services to generate profits, which in turn, are required to be redistributed as bonuses.[2] In line with Littlechild and Owen [1973], such firms can be mod-

model of queueing problems.

There are also papers that present modifications or extensions of Shapley value by either altering the underlying axioms or by imposing structures on the opportunities of cooperation.[6] One of most interesting of such modifications in recent times, is the concept of `*egalitarian Shapley values*', which are convex mixtures of Shapley value and the equal division of the grand coalition among all players. Two notable papers discussing these egalitarian Shapley values are Casajus and Huettner [2014] and Casajus and Huettner [2013] and van den Brink et al. [2013]. van den Brink et al. [2013] completely characterized this class using conventional efficiency, linearity, anonymity and weak monotonicity (a condition that is weaker than our marginal monotonicity). Casajus and Huettner [2014] characterize these solutions as the only ones that satisfy conventional efficiency, symmetry and weak monotonicity.

Our paper, too, looks for a new solution for a cooperative game that satisfies *extended efficiency* (instead of conventional efficiency) along with the standard axioms of symmetry and monotonicity. As argued above, our second solution that is developed using marginal monotonicity, presents an extension of Young's result to the idea of extended efficiency. As noted in Casajus and Huettner [2014], there are only two other such generalizations of Young's result in the transferable utility framework: Nowak and Radzik [1995] and De Clippel and Serrano [2008]. The former relaxes the symmetry assumption to present a characterization of weighted Shapley values, while the latter presents a extension of Shapley value to cooperative games with externalities (requiring the primitive to be partition function instead of characterisitic function).

With respect to our egalitarian value obtained using coalitional monotonicity, two relevant papers are: van den Brink [2007] and Moulin [1987]. The latter paper characterizes the solution that equally divides grand coalitional worth among all players in a setting where players are identified by heterogeneous opportunity costs. The former paper explores connections between the equal division of grand coalitional worth and the null player property of Shapley [1953]. It provides characterizations of this value using a modification of this null

Model

Consider a set $N = \{1, 2, \ldots, n\}$ where $n \geq 2$. For any set $S \subseteq N$, let (S) be the set of all possible non-empty subsets of S. Define a transferable utility cooperative game to be a pair (N, v) where N is the set of players and $v : (N) \cup \emptyset \mapsto \mathbb{R}$ is a characteristic function that assigns to each possible coalition in the game a real valued worth and $v(\emptyset) := 0$. Let $V(N)$ denote the class of all such characteristic functions that can be defined on the set (N), and define $\{(N, v)\}_{v \in V(N)}$ to be the class of all possible games that can be defined on the player set N. Note that we do not impose any superadditivity restriction on the set of functions $V(N)$.

Our objective is to obtain a solution (that is, a value distribution across players) for each possible game so that a society of players can make an informed choice on which games to play. That is, we seek to obtain a solution $: V(N) \mapsto \mathbb{R}^N$. In this paper, we require such

Our third axiom requires that the solution satisfy `coalitional monotonicity' in the sense that the value assigned to any player i should not decrease, when the underlying characteristic function v changes to any other function v' in a manner such that worths of all coalitions containing i increases. That is, value assigned to an agent should increase if profitability of all cooperative groups containing her improves.

Definition 3 $\psi(\cdot)$ *satisfies coalitional monotonicity (C-MON) if and only if for all* $v, v' \in V(N)$, *and all* $i \in N$,

$$[v(S \cup \{i\}) \geq v'(S \cup \{i\}), \; \forall S \subseteq N \setminus \{i\}] \Rightarrow [\psi_i(v) \geq \psi_i(v')].$$

Our fourth axiom presents an alternative notion of monotonicity which requires that a solution satisfy `marginal monotonicity' (as proposed in Young [1985]). This idea of monotonicity requires that the value assigned to any player i should not decrease, when the underlying characteristic function changes in a manner such that marginal contributions of i to all groups containing her, increase. As in Shapley [1953], we quantify such a marginal contribution of a player i to any team $S \subseteq N$, by the difference $c_v^i(S) := v(S) - v(S \setminus \{i\})$ with the convention that $c^i($0p in Ym52 Tf 4.552 0 Td [(v)] TJ /F52 7.97 nat3

Definition 3 v(S

Proof: See Appendix.

Note that Theorem 1 sums up the averages of coalitional worths to obtain the value for an individual player. Therefore, for a two player game $(\{1, 2\}, v)$, Theorem 1 implies a solution:

$$\phi_1(v) = \frac{v(12)}{2} + v(1), \quad \phi_2(v) = \frac{v(12)}{2} + v(2);$$

while for a three player game $(\{1, 2, 3\}, v)$, it proposes a solution:

$$\begin{aligned} \phi_1(v) &= \frac{v(123)}{3} + \frac{v(12)}{2} + \frac{v(13)}{2} + v(1); \\ \phi_2(v) &= \frac{v(123)}{3} + \frac{v(12)}{2} + \frac{v(23)}{2} + v(2); \\ \phi_3(v) &= \frac{v(123)}{3} + \frac{v(13)}{2} + \frac{v(23)}{2} + v(3). \end{aligned}$$

Main Result

Observe that averaging of coalitional worths prior to its addition in the solution proposed by Theorem 1, lends an egalitarian character to the implied value distribution. However, it is unlikely that all members of a team put in equal amounts of efforts in generating team profits or worths. One way to account for any difference in effort or productivity of a member of a group, is to compute her marginal contribution to the team as in Shapley [1953]. The following theorem presents our main result, which states the implication of using marginal monotonicity instead of coalitional monotonicity. We find that ϕ is the unique solution that satisfies extended efficiency, symmetry and marginal monotonicity.

Theorem 2

satisfies EFF, we first note that for any $v \in V(N)$, any i and any $t = 1, \dots, n$,

$$\sum_{i \in N} \sum_{\substack{S \in \Omega(N); \\ |S| = t; i \in S}} c_v^i(S) = t \sum_{\substack{S \in \Omega(N); \\ |S| = t}} v(S$$

In case (ii), define a set $\{S_1, S_2, \ldots, S_t\} := \{S \in (N) | v(S) \neq 0, S \neq N\}$. Since $(v) = k + 1$, such a set is well defined. Further, define $E := \cap_{r=1}^{t} S_r$ and note that $i \in E$ by construction.[7] If $|E| > 1$, then fix any $j \neq k \in E$, and consider the bijection $\pi^{jk} : N \mapsto N$ such that $\pi^{jk}(j) = k$, $\pi^{jk}(k) = j$, and $\pi^{jk}(l) = l$ for all $l \in N \setminus \{j, k\}$. By construction, for any $T \subseteq N$, if $E \subseteq T$, then $\pi^{jk}(T) = T$ which implies that $\pi^{jk}v(T) = \pi^{jk}v(\pi^{jk}(T)) = v(T)$;[8] or else (that is, if E is not a subset of T), $\pi^{jk}v(T) = v(T) = 0$. Therefore, by SYM, it follows that whenever $|E| > 1$, $\phi_j(v) = \phi_k(v)$, $\forall \{j, k\} \subseteq E$. Thus, EFF implies that for all $j \in E$, $\phi_j(v) = ^1$

while for a three player game $(\{1;2;3\};v)$, it proposes a solution:

$$\begin{aligned}
\phi_1(v) &= \frac{v(123)-v(23)}{3} + \frac{2\{[v(13)-v(3)]+[v(12)-v(2)]\}}{3} + \frac{7}{3}v(1)\\
\phi_2(v) &= \frac{v(123)-v(23)}{3} + \frac{2\{[v(13)-v(3)]+[v(12)-v(2)]\}}{3} + \frac{7}{3}v(2)\\
\phi_3(v) &= \frac{v(123)-v(12)}{3} + \frac{2\{[v(13)-v(3)]+[v(12)-v(2)]\}}{3} + \frac{7}{3}v(3).
\end{aligned}$$

Note how, unlike Theorem 1, Theorem 2 requires that the weights given to marginal contribution of any player to groups containing her, decrease as the group sizes increase. So the least weight is given to the marginal contribution to the grand coalition, while the maximum weight is given to the singleton coalition (that is, what the player can do alone).

Note that a difficult feature of functional form of the value $\phi(.)$ presented in Theorem 2 is that the coefficients $\alpha^n_1; \alpha^n_2; \ldots; \alpha^n_n$ are defined in a recursive manner. The following corollary presents a simpler functional formulation of the β_t values.

Corollary 1 *For any $t = 1;\ldots;n$,*

$$\alpha^n_t = (n-t)!(t-1)!\sum_{k=0}^{t-1}\binom{n}{k}.$$

Proof: We prove this result by induction. Note that $\alpha^n_1 = \frac{n!+(1-1)\alpha^n_0}{n-1+1} = (n-1)!\,0!\binom{n}{0}$. Now suppose that for all $m \in \mathbb{N}$, $\alpha^n_m = (n-m)!(m-1)!\sum_{k=0}^{m-1}\binom{n}{k}$. Then, by Theorem 2,

$$\begin{aligned}
\alpha^n_{m+1} &= \frac{n!+(m+1-1)\alpha^n_m}{n-(m+1)+1}\\
&= \frac{n!+m(m-1)!(n-m)!\sum_{k=0}^{m-1}\binom{n}{k}}{n-m}\\
&= m!(n-m-1)!\left\{\frac{n(n-1)\ldots(n-m+1)}{m!}+\sum_{k=0}^{m-1}\binom{n}{k}\right\}\\
&= \{(m+1)-1\}!\{n-(m+1)\}!\sum_{k=0}^{(m+1)-1}\binom{n}{k}
\end{aligned}$$

and so, the result follows.

Therefore, in light of Corollary 1, for any $i \in N$ and $v \in V(N)$, ϕ_i can be rewritten as follows:

$$\phi_i(v) := \sum_{S\in\mathcal{P}(N)} \beta_s c^i_v(S)$$

where for all $t = 1;\ldots;n$, $\beta_t := \frac{\alpha^n_{n-t+1}}{n!} = \frac{1}{\binom{n-1}{t-1}}\sum_{k=0}^{n-t}\frac{1}{n-k}\binom{n-1}{k}$ and $s := |S|;\forall S \in \mathcal{P}(N)$. Thus, Corollary 1 allows us to represent $\phi_i(\cdot)$ as a linear combination of marginal contribution of player i, with the weights being given by $_S$.

The following example provides a contrast between the two values presented by Theorems 1 and 2, by applying them to the contentious, but relevant, problem of bonus distribution

	1	2	3
(*Theorem 1*)	$\frac{115}{3}$	$\frac{130}{3}$	$\frac{145}{3}$
(*Theorem 2*)	$\frac{100}{3}$	$\frac{130}{3}$	$\frac{160}{3}$

Table 3: *Bonus distributions.*

assigns worth of any group of players $S \subseteq N$ to be $w(S) := \sum_{T \subseteq S} v($

Most importantly, however, this alternate manner of constructing a characteristic function may be socially *unacceptable* in a practical setting, as the value generated by a group of players gets attributed to a larger set. That is, for a simple two player game $(\{1,2\}, v)$: the marginal contribution of 1 is now $v(2) + v(12)$, which is unlikely to be acceptable to the player 2, who finds that working harder on her own enhances the marginal contribution of her competitor within the organization. Similarly, player 1 would find it difficult to accept such accounting procedure where her marginal contribution vector depends on the individual performance of her competitor. Hence, from an application perspective to real life problems, potentially of great importance in a country's economy, our approach of constructing a characteristic function is more useful.[9]

Conclusion

In this paper, we formalize a novel notion of extended efficiency to conceptualize the no-wastage condition in settings where the traditional notion of efficiency is not applicable. Such settings are those where the members of a society work in sub-groups to generate resources for the society; like the gross national product of a nation being generated by various cooperative enterprises among sub-groups of her citizens. Unlike the conventional efficiency axiom of cooperative game theory literature, this axiom requires that a solution to a game assign individual values that sum up to equal the sum of worths of all possible coalitions.

We use this novel axiom, along with the standard monotonicity and symmetry axioms to characterize a new solution for cooperative games which, in the spirit of Young [1985], presents an extension of Shapley value to these practical settings.

Appendix

Independence of Axioms

Theorem 1

For simplicity of exposition, consider a 2-player game $(N = \{1, 2\}, v)$. Clearly there are three possible coalitions: $\{1\}$, $\{2\}$ and $\{1, 2\}$. Consider the following solutions:

$\phi_1(v) = v(\{1\}) + 0.8v(\{1,2\})$, $\phi_2(v) = v(\{2\}) + 0.2v(\{1,2\})$. It is easy to see that this solution satisfies EFF and C-MON. However, if the agent labels were interchanged, the individual values would not get interchanged for all possible $v(.)$ - implying that this rule does not satisfy SYM.

$\phi_1(v) = v(\{1\}) + \frac{v(\{1,2\})}{4}$, $\phi_2(v) = v(\{2\}) + \frac{v(\{1,2\})}{4}$. It is easy to see that this solution satisfies C-MON and SYM. However, for any $v(.)$, $\phi_1(v) + \phi_2(v) = v(\{1\}) + v(\{2\}) + \frac{v(\{1,2\})}{2}$, and so, this rule does not satisfy EFF.

$\phi_1(v) = v(\{1\}) + \frac{v(\{1\})v(\{1,2\})}{v(\{1\})+v(\{2\})}$, $\phi_2(v) = v(\{2\}) + \frac{v(\{2\})v(\{1,2\})}{v(\{1\})+v(\{2\})}$. It is easy to see that this rule satisfies EFF and SYM. However, consider two characteristic functions, $w(.)$ and $w'(.)$ such that $w(\{1,2\}) = w'(\{1,2\})$, $w(\{1\}) = w'(\{1\})$ and $w(\{2\}) > w'(\{2\})$. It is easy to see that $\phi_1(w) < \phi_1(w')$ even though 1's coalitional worths in the groups $\{1\}$ and $\{1,2\}$ remain unchanged across characteristic functions w and w'. Note that, by C-MON,

$$[w(\{1\}) = w'(\{1\}); w(\{1,2,\}) = w'(\{1,2,\})] \implies \phi_1(w) = \phi_1(w');$$

and so, this solution violates C-MON.

Theorem 2

As before, for simplicity of exposition, we consider a 2 player game $(N = \{1,2\}, w)$ with three possible coalitions: $\{1\}$, $\{2\}$ and $\{1,2\}$. Consider the following solutions:

$\phi'_1(v) = 0.75v(\{1\}) + 0.25(v(\{1,2\}) - v(\{2\}))$, $\phi'_2(v) = 0.75v(\{2\}) + 0.25(v(\{1,2\}) - v(\{1\}))$. It is easy to see that this solution satisfies M-MON and SYM. However, $\phi'_1(v) + \phi'_2(v) = 0.5[v(\{1,2\}) + v(\{1\}) + v(\{2\})]$, and so, it does not satisfy EFF.

Fix a small enough $\epsilon > 0$, and consider $\phi'_1(v) = 1.5v(\{1\}) + 0.5(v(\{1,2\}) - v(\{2\})) + \epsilon$, $\phi'_2(v) = 1.5v(\{2\}) + 0.5(v(\{1,2\}) - v(\{1\})) - \epsilon$. It is easy to see that this rule satisfies EFF and M-MON but does not satisfy SYM (as an interchange of agent labels would not lead to interchange in individual values).

$\phi'_1(v) = v(\{1\}) + \frac{v(\{2\})v(\{1,2\})}{v(\{1\})+v(\{2\})}$, $\phi'_2(v) = v(\{2\}) + \frac{v(\{1\})v(\{1,2\})}{v(\{1\})+v(\{2\})}$. It is easy to see that this solution satisfies EFF and SYM. However, consider two characteristic functions v and v' such that $v(\{2\}) > v'(\{2\})$ and $v(S) = v'(S)$ when $S \in \{\{1\},\{1,2\}\}$. This means that $c^1_v(\{1\}) = c^1_{v'}(\{1\})$, and $c^1_v(\{1,2\}) < c^1_{v'}(\{1,2\})$. Therefore, M-MON requires that $\phi'_1(v) \leq \phi'_1(v')$. However, by construction, $\phi'_1(v) > \phi'_1(v')$, and so, it follows that this solution violates M-MON.

Proof of Theorem 1

Define for all $i \in N$ and all $v \in V(N)$, $\phi_i(v) := \sum_{S \in \Pi(N); i \in S} \frac{v(S)}{|S|}$.[10]

It can easily be checked that $\phi_i(v)$ satisfies EFF, SYM and C-MON, and so, the proof of sufficiency follows. To prove necessity, fix any solution $\phi(.)$ satisfying EFF, SYM and C-MON, and any $i \in N$. Now consider the partition of $V(N)$ into the set $P := \{V^0, V^2, \ldots, V^{|\Pi(N)|}\}$ such that for all $k = 0, \ldots, |\Pi(N)|$, V^k is the set of characteristic functions such that there are exactly k teams in $\Pi(N)$ who have posted zero profit/worth. It can easily be seen that: (i) by construction $V(N) = \mathbb{R}_+^{|\Pi(N)|}$, (ii) for all $k \neq l \in \{1, \ldots, n\}$, $V^k \cap V^l = \emptyset$, and $\cup_{k=0}^{n} V^k = \mathbb{R}_+^{|\Pi(N)|}$. Hence, P is well defined.

Now fix any characteristic function $v^{|\Pi(N)|} \in V^{|\Pi(N)|}$, any $i \neq j$, and any permutation $\pi^{ij} : N \mapsto N$ such that $\pi^{ij}(i) = j; \pi^{ij}(j) = i$. Note that by SYM, $\phi_i(v^{|\Pi(N)|}) = \phi_j(v^{|\Pi(N)|})$. Hence, EFF implies that $n\phi_i(v^{|\Pi(N)|}) = 0 \implies \phi_i(v^{|\Pi(N)|}) = 0; \forall i \in N$. Now suppose that for some $l \in \{1, \ldots, |\Pi(N)|\}$, (a) $v^l \in V^l \implies \phi_i(v^l) = \phi_i(v^l); \forall i \in N$. Now consider any $v^{l-1} \in V^{l-1}$, and define $\tilde{N}(v^{l-1}) := \{i \in N | \forall S \in \Pi(N); i \notin S \implies v^{l-1}(S) = 0\}$. Thus $\tilde{N}(v^{l-1}) \subseteq N$ is the set of agents i such that any team $S \in \Pi(N)$ not containing i, has zero worth in v^{l-1}. Therefore, if $\tilde{N}(v^{l-1}) = N$, then for any $S \in \Pi(N)$, $S \neq N \implies v^{l-1}(S) = 0$, and so, $v^{l-1}(N) > 0$ (as $l - 1 < |\Pi(N)|$ by construction). Now, as before, any $i \neq j \in N$, and any permutation π^{ij} with $\pi^{ij}(i) = j$ and $\pi^{ij}(j) = i$, $\pi^{ij} v^{l-1} = v^{l-1}$, and so, by SYM, $\phi_i(v^{l-1}) = \phi_j(v^{l-1})$. Hence, EFF implies that $\phi_i(v^{l-1}) = \phi_i(v^{l-1})$. This establishes the result for the case where $\tilde{N}(v^{l-1}) = N$.

Now, if $\tilde{N}(v^{l-1}) \subsetneq N$, then for any $i \notin \tilde{N}(v^{l-1})$, choose a $T^i(v^{l-1}) \in \Pi(N)$ such that $i \notin T^i(v^{l-1})$ and $v^{l-1}(T^i(v^{l-1})) > 0$. Note that by construction of $\tilde{N}(v^{l-1})$, the set $T^i(v^{l-1})$ is well defined. Construct a characteristic function $\tilde{v}_i^l \in V^l$ where for all $S \in \Pi(N)$, $S \neq T^i(v^{l-1}) \implies \tilde{v}^l(S) = v^{l-1}(S)$ and $\tilde{v}^l(T^i(v^{l-1})) = 0$. By supposition (a) and C-MON, $\phi_i(v^{l-1}) = \phi_i(\tilde{v}^l) = \phi_i(\tilde{v}^l)$ for all $i \notin \tilde{N}(v^{l-1})$. This establishes the result for the case where $\tilde{N}(v^{l-1}) = \emptyset$. Further, if $|\tilde{N}(v^{l-1})| = 1$, that is, supposing $\tilde{N}(v^{l-1}) = \{l\}$, by EFF, $\phi_l(v^{l-1}) = \sum_{S \in \Pi(N)} v(S) - \sum_{i \notin \tilde{N}(v^{l-1})} \phi(v^{l-1})$ which equals $\phi_l(\tilde{N}(v^{l-1}))$, because as argued in proof of sufficiency above, $\phi(.)$ satisfies EFF.

Now, to establish the result for the only remaining possibility where $0 < |\tilde{N}(v^{l-1})| < n$, note that by construction, for any $S \in \Pi(N)$, $v^{l-1}(S) > 0 \implies \tilde{N}(v^{l-1}) \subseteq S$. Therefore, for any $i \neq j \in \tilde{N}(v^{l-1})$, and any permutation π^{ij} such that $\pi^{ij}(i) = j; \pi^{ij}(j) = i$, $\pi^{ij} v^{l-1} = v^{l-1}$, and so, by SYM, $\phi_i(v^{l-1}) = \phi_j(v^{l-1})$. Therefore, EFF implies that for all $i \in \tilde{N}(v^{l-1})$, $|\tilde{N}(v^{l-1})| \phi_i(v^{l-1}) = \sum_{S \in \Pi(N)} v^{l-1}(S) - \sum_{j \notin \tilde{N}(v^{l-1})} \phi_j(v^{l-1}) = \sum_{S \in \Pi(N)} v^{l-1}(S)$

[10] The proof technique resembles a similar result is proved in Mukherjee et al. [2020].

Casajus, A. and Yokote, K. (2017). Weak di erential marginality and the Shapley value. *Journal of Ecoalue.*