Indian Institute of Management Calcutta

Working Paper Series

WPS No 856 /January 2021

A Polyhedral Study of Generalized Assignment Problem with Demand Constraints

Prasenjit Mandal*
Assistant Professor, Operations Management Group
Indian Institute of Management Calcutta
Joka, D H Road, Kolkata 700104, INDIA
e-mail: prasenjitm@iimcal.ac.in

*Corresponding Author

Indian Institute of Management Calcutta, Joka, D.H. Road, Kolkata 700104

URL: http://facultylive.iimcal.ac.in/workingpapers

A Polyhedral Study of Generalized A ignment Pro lem with Demand Con traint

1 Introduction

The traditional Generalized Assignment Problem (GAP) is a classical NP-hard discrete o timization rob-lem. It consists of minimizing the assignment costs of a set of jobs to a set of machines while satisfying the ca acity constraints. It is one of the most widely addressed roblems in the integer rogramming and combinatorial o timization literature (Cattrysse and Van Wassenhove, 1992).

The ur ose of this a er is to study a roblem similar to the GAP where a set of agents with limited roficiency are assigned to a set of jobs to satisfy their demands. The demand constraints are ty ically the well-known kna sack inequalities in the form of greater-than-or-equal-to ty e constraints. Like the GAP, an agent can be assigned to one job only. I assume that the cost of assignment is ro ortional to the roficiency of the agent. I refer to this roblem as generalized assignment roblem with demand constraints (GAPD). Hence, it is a variant of the GAP.

GAPD has numerous real life a lications and it may also a ear as a sub- roblem in several other roblems. Although, I started with a roblem that considers assignment of agents to jobs, roblems with similar structures arise in many other real life scenarios. I rovide a few such exam les here. In a software develo ment firm, managers often estimate the man-hour requirements for the ongoing rojects and allocate a grou of software rofessionals in form of teams to different rojects to meet the requirements. Also, GAPD a ears as a sub- roblem to staff scheduling and rostering roblem where a firm constructs work timetables for its staff to satisfy the demand for goods or services. The a lication areas of staff scheduling and rostering include health care systems, trans ortation services such as airlines and railways, emergency services such as olice, ambulance and fire brigade, call centres, and other service firms like hotels, restaurants and retail assigned to job j N. I denote d_j ≥

$\sum_{j \in N} \sum_{k \in S_j} c_k \le \alpha$. Now, any data instance I of the $3-$PARTITION roblem can be seudo- olynomially transformed, without loss of generality, into an equivalent instance $\hat{I}$ of the restricted GAPD (i.e., a case of multi le kna sack roblem) by setting $d_j = B$ for $j \in N$, $c_k = 1$ for $k \in M$ and $\alpha = |M|$ (Martello, 1990). As a $3-$PARTITION roblem is strongly NP-hard, the restricted GAPD is also NP-hard (Garey and Johnson, 1979). Hence, the GAPD as a generalization of the restricted GAPD must also be NP-hard. □

2.1 Individual Cover Inequalities

GAPD has a s ecial structure. The roblem consists of $|N|$ number of greater-than-equal-to ty e of kna sack constraints. Let, $P_{KP}(j)$ denotes the kna sack olyto e corres onding to job $j \in N$. Then,

$$P_{KP}(j) = \left\{ \sum_{k \in M} a_{jk} x_{jk} \ge d_j | x_{jk} \in \{0,1\}, k \in M \right\}, \quad j \in N.$$

The kna sack olyto e $P_{KP}(j)$ is a relaxation of X_{GAPD}. Cover inequalities were introduced by Balas Balas (1975), Hammer et al. Hammer et al. (1975) and Balas et al. Balas and Zemel (1978) for a kna sack olyto e. Later, Gottlieb and Rao Gottlieb and Rao (1990b) also derived the individual cover inequalities for GAP. Here I resent similar inequalities for the $P_{KP}(j)$.

Definition 2.1. A set $C_j \subseteq M$, $j \in N$ and $\bar{C}_j := M \setminus C_j$. C_j is an individual cover for $j \in N$ if

$$\sum_{k \in \bar{C}_j} a_{jk} < b_j.$$

If C_j is a cover for job $j \in N$, then $\bar{C}_j$ is also defined *is v lid for* P_{GAPD}.

Next, I introduce the extended individual cover to obtain stronger inequalities. For a minimal individual cover C_j, let $a_j := \max_{k \in C_j} a_{jk}$ and $E(C_j) = \{k \in \bar{C}_j | a_{jk} \geq a_j\}$. Then, the following set of inequalities are referred as extended individual cover inequalities:

$$\sum_{k \in C_j \cup E(C_j)} x_{jk} \geq 1 + |E(C_j)| \tag{5}$$

Similar to Gottlieb and Rao Gottlieb and Rao (1990b), I also derive the set individual $(1, k_j)$-configuration inequalities for each job.

Definition 2.3. For each $j \in N$, a set $M_j^{\square} \cup \{z\}$ is a $(1, k_j)$-configuration if $M_j^{\square} \subseteq M$, $|M_j^{\square}| = m_j^{\square}$ and $z \in M \setminus M_j^{\square}$ are such that

(i) $\sum_{k \in M \setminus M_j^{\square}} a_{jk} \geq d_j$,

(ii) $K_j \cup \{z\}$ is a minimal cover for each $K_j \subseteq M_j^{\square}$ with $|K_j| = k_j$ where k_j is an integer satisfying $2 \leq k_j \leq m_j^{\square}$ (i.e., elements in $M \setminus (K_j \cup \{z\})$ can't satisfy the demand d_j).

Proposition 3. *The individu l* $(1, k_j)$*- on gur tion inequ lity*

$$(r_j - k_j + 1)x_{jz} + \sum_{k \in R_j} x_{jk} \geq (r_j - k_j + 1) \tag{}$$

is v lid for P_{GAPD}*, where* $R_j \subseteq M_j^{\square}, |R_j| = r_j$ *s tisfying* $k_j \leq r_j \leq m_j^{\square}$.

If $k_j = m_j^{\square}$, I observe that the individual $(1, k_j)$-configuration is a individual minimal cover.

2.2 Multiple Cover Inequalities

In this section, I restrict my attention to inequalities that consider multi le jobs. Next in Pro osition 4, I resent several classes of valid inequalities corres onding to a subset of jobs.

Proposition 4. *( ) For some job* $p \in N$*, let* $S \subseteq M$ *be set of gents su h th t* S *is over, i.e.,* $\sum_{k \in \bar{S}} a_{pk} <$ d_p*. Let,* $\underline{k}_p = \arg\min_{k \in \bar{S}} a_{pk}$*. There doesn't exist ny gent* $v \in S$*, su h th t* $\sum_{k \in \bar{S} \setminus \{\underline{k}_p\}} a_{pk} + a_{pv} \geq d_p$*, i.e., substituting ny gent from set* S *for the gent in* $\bar{S}$ *with minimum pro ien y is not enough to s tisfy the dem nd* d_p.

(b) For nother job $l \in N, l = p$*, let* $\bar{T} \subseteq S$ *be set of gents su h th t* $\bar{T} \cup \{s\}$ *is n nti- over for ll* $s \in \bar{S}$*, i.e.,* $\sum_{k \in \bar{T}} a_{lk} + a_{ls} < b_l$*. quiv lently, for ll gent* $s \in \bar{S}$*, the set of gents* $T \setminus \{s\}$ *is denoted to be over for job* l*, where* $T = M \setminus \bar{T}$.

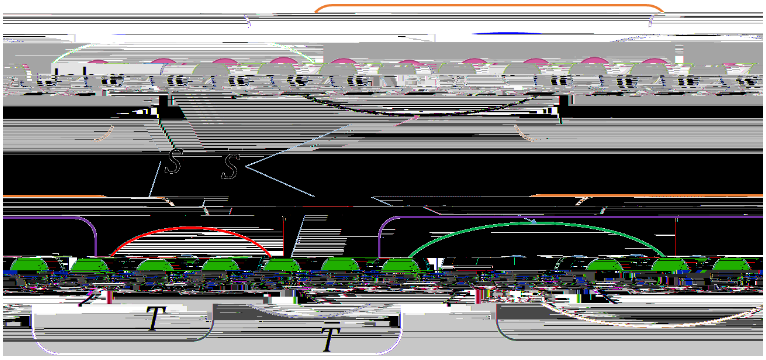

Figure 1: Multiple Cover Inequality

( ) Also, there doesn't exist any agent $t \in S \setminus \bar{T}$*, such that the set of agents* $\bar{T} \cup \{t\}$ *satisfy the demand* d_l*, i.e.* $\sum_{k \in \bar{T} \cup \{t\}} a_{lk} \geq d_l$.

Then the following inequality is valid for the P_{GAPD} *polytope:*

$$\sum_{k \in S} x_{pk} + \sum_{k \in T \setminus \bar{S}} x_{lk} \geq 3.$$

Proof. To prove the proposition, I consider three non-trivial cases.

Case 1: For a job p, let $x_{pk} = 1, k \in \bar{S}$, and for job l, let $x_{lk} = 1, k \in \bar{T}$. In that case, at least 1 additional resource is required to complete job p, whereas at least 2 additional resources are required to complete job l, i.e., $\sum_{k \in S} x_{pk} \geq 1$ and $\sum_{k \in T \setminus \bar{S}} x_{lk} \geq 2$.

Case 2: For a job p and for an agent $s \in \bar{S}$ let $x_{pk} = 1, k \in \bar{S} \setminus \{s\}$; whereas for job l, let $x_{lk} = 1, k \in \bar{T}$ and $x_{ls} = 1$. From Proposition 1(a), I know that $\sum_{k \in \bar{S} \setminus s} a_{pk} + a_{pv} < d_p$ for all agent $v \in S$; hence, at least 2 additional resources are required to complete job p, i.e., $\sum_{k \in S} x_{pk} \geq 2$. From Proposition 1(b), I know that $\bar{T} \cup \{s\}$ is an anti-cover and from Proposition 1(c), I know that $\sum_{k \in \bar{T} \cup \{t\}} a_{lk} < d_l$ for any agent $t \in S \setminus \bar{T}$. Hence, at least 1 additional resource is required to complete job l, i.e., $\sum_{k \in T \setminus \bar{S}} x_{lk} \geq 1$.

Case 3: For a job p and for any two agents $s_1, s_2 \in \bar{S}$ let $x_{pk} = 1, k \in \bar{S} \setminus \{s_1, s_2\}$ and for an agent $t \in \bar{T}$ (i.e., $t \in S$ as $S \supseteq \bar{T}$) , $x_{pt} = 1$; whereas for job l, let $x_{lk} = 1, k \in \bar{T} \setminus \{t\}$ and $x_{l,s_1} = 1, x_{l,s_2} = 1$. In that case, from Proposition 1(a), it can be easily shown that at least 1 additional resource is required to complete the job p, i.e., $\sum_{k \in S} x_{pk} \geq 2$. From Proposition 1(b) and Proposition 1(c), it can also be easily shown that at least one additional resource is required to complete job l, i.e., $\sum_{k \in T \setminus \bar{S}} x_{lk} \geq 1$.

For all these three non-trivial cases presented above, I need exactly 3 agents to complete both the jobs p and l. For all other trivial cases, it can easily shown that the minimum number of agents required to both the jobs p and l are at least 3. Hence, the inequality $\sum_{k \in S} x_{pk} + \sum_{k \in T \setminus \bar{S}} x_{lk} \geq 3$ is a valid one. It completes the roof of the ro osition. □

Example 1. *Let us onsider n ex mple with* 2 *jobs nd* 5 *gents. The onstr ints to the problem is given by:*

$$4x_{11} + 3x_{12} + 5x_{13} + 4x_{14} + 3x_{15} \geq 7$$

$$3x_{11} + 4x_{12} + 5x_{13} + 2x_{14} + 3x_{15} \geq 8$$

Let, $p = 1$ *nd* $\bar{S} = \{1\}$*, then over set* $S = \{2, 3, 4, 5\}$*. Set* $\{1\}$ *be nti- over for job* 1 *s* $4 < 7$ *(doesn't s tisfy dem nd). The inequ lity* $x_{12} + x_{13} + x_{14} + x_{15} \geq 1$ *is over inequ lity for job* 1.

Let, $l = 2$ *nd* $\bar{T} = \{4\}$*, then* $T = \{1, 2, 3, 5\}$*. For ll* s $\bar{S} = \{1\}$*,* $\bar{T}$ $\{s\}$ *is n nti- over. The over for job* 2 *is the set* $T \setminus \{s\}$*,* s $\bar{S}$*. Then, the inequ lity* $x_{12} + x_{13} + x_{15} \geq 1$ *is over inequ lity for job* 2*. A over inequ lity onsidering multiple jobs is given by:*

$$\sum_{k=2}^{5} x_{1k} + \sum_{k \in \{2,3,5\}} x_{2k} \geq 3.$$

The set of ll fe sible integer points re given below. The inequ lity bove s tis es ll the fe sible integer points. At the s me time, ple se he k th t, for $l = 2$*, if* T = ==

$$\sum_{k \quad S} x_{pk} + \sum_{j \quad W \setminus \{p\}} \sum_{k \quad T_j \setminus \bar{S}} x_{jk} \geq |W| + 1.$$

The roof for Corollary 1 is essentially the same as for Pro osition 4. Next in Pro osition 5, I resent another variant of multi le cover inequality.

Proposition 5. *( ) For some job* $p \quad W$, $W \quad N$ *nd set of jobs* $\bar{C} \quad M$, $|\bar{C}| = c$ *is su h th t* $\bar{C}$ *is n nti- over (* $C = M \setminus \bar{C}$ *is over), i.e.,* $\sum_{k \quad \bar{C}} a_{pk} < d_p$. *Let,* $\underline{k}_p = \arg\min_{k \quad \bar{C}} a_{pk}$. *There doesn't exist ny gent* $s \quad C$*, su h th t* $\sum_{k \quad \bar{C} \setminus \{\underline{k}_p\}} a_{pk} + a_{ps} \geq d_p$*, i.e., substituting ny gent from set* C *for the gent in* $\bar{C}$ *with minimum pro ien y is not enough to s tisfy the dem nd* d_p.

(b) for e h job $j \quad W \setminus \{p\}$ *, there exists set of gents* $\bar{T}_j \quad C$ *nd*

$$\bar{C}_j = \left(k \quad \bar{C} \cup \bar{T}_j \right) \{k\} \text{ is n nti over for } k$$

$$\frac{1}{c_m}\sum_{k \in C} x_{pj} + \sum_{j \in W \setminus \{p\}} \left(\frac{1}{c_j} \sum_{u \in \bar{C}_j} \sum_{k \in T_j} \left(x_{jk} - x_{ju} \right) \right) + \frac{c_m - 1}{c_m} c$$

$$\geq \frac{1}{c_m} + \sum_{j \in W \setminus \{p\}} \frac{c_j}{c_j} + \frac{c_m - 1}{c_m} \sum_{k \in \bar{C}} \sum_{j \in W} x_{jk}$$

$$= \frac{1}{c_m} + |W \setminus \{p\}|$$

$$\frac{1}{c_m}\sum_{k \in C} x_{pj} + \frac{c_m - 1}{c_m}\sum_{k \in C} x_{pk} + \sum_{j \in W \setminus \{p\}} \left[\frac{1}{c_j} \sum_{u \in \bar{C}_j} \left(\sum_{k \in T_j} x \right. \right.$$

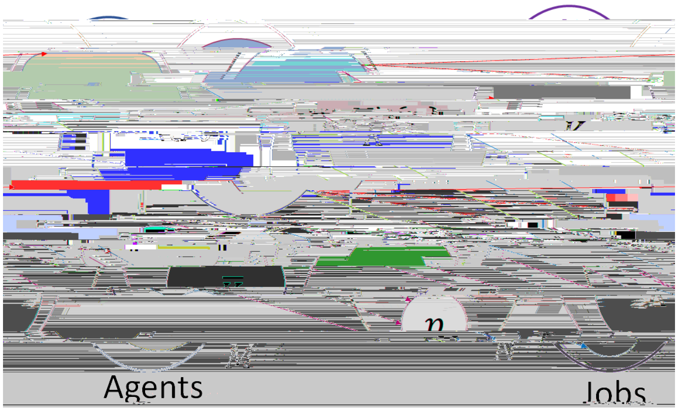

Figure 2: Multi le Cover Inequality

By definition of the flow cover, if the agents in the set $\bar{K}$ M is already assigned to job p N, then the residual demands that are required to fulfil are,

$$\sum_{k \quad K} a_{pk} x_{pk} \geq \lambda.$$

If any agent $_1$ K is assigned to job p by kee ing all the agents in $K \setminus \{ _1\}$ left unassigned, then the minimum flow required to fulfil the demand d_p are

$$\sum_{k \quad K \setminus \{ _1\}} a_{pk} x_{pk} = \max\{\lambda - a_{p\ _1}, 0\} = \lambda - \min\{\lambda, a_{p\ _1}\}.$$

I extend it further by induction. If any two agents $_1, _2$ K are assigned to job p by kee ing all the agents in $K \setminus \{ _1, _2\}$ left unassigned, then the minimum flo;85.504-30.3-

$$\sum_{k \in K} a_{pk} x_{pk} \geq \lambda - \sum_{k \in K} \min\{\lambda, a_{pk}\} \left(1 - \sum_{j \in N \setminus \{p\}} x_{jk}\right)$$

$$\sum_{k \in K} a_{pk} x_{pk} + \sum_{k \in K} \min\{\lambda, a_{pk}\} \left(1 - \sum_{j \in N \setminus \{p\}} x_{jk}\right) \geq \lambda,$$

so the inequality is valid. □

3 Conclusion and ecommendations for Future esearch

This a er establishes several valid inequalities to solve the GAPD effectively. Thus, I study the olyhedral ro erties of the convex hull of the GAPD which com rises of a set of greater-than-equal-to ty es of kna sack inequalities (each kna sack corres onds to a job) with SOS constraints. The GAPD a ears as a