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# Sensitivity of Centrality and Centralization Measures to the Level of Decentralization in the Network Structure

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global economic developments and cultural differences between countries. Recent trend in globalization attempts to exploit decentralized human capital and knowledge base by collaborative works while retaining the administrative control by a central unit. In response to such opportunities, the organizations are tending to follow a more decentralized structure. A reliable measure of the level of decentralization in such a network would play a key role in assessing the impact of decentralization of work centers, governance, transportation and communication infrastructure on economic development, effectiveness of social policy implementations, supply chain management, project management, traffic management, and knowledge transfer in a social network (Bardhan 2002).

The social structure in different subgroups of large product development teams may be different due to regional influence or organizational culture. This often leads to intergroup coordination and communication problems that may cause hindrances to product delivery or meeting deadlines. Therefore, organizations have to do a trade-off between the level of centralization and cost. (Gupta & Govindarajan 1991) emphasizes that the knowledge flow patterns among different subsidiaries within the same MNC located in different locations can differ as reflected in the way a mix of formal and informal administrative control mechanisms are used to shape the actions of various subsidiaries and for each type of transactions, different subsidiaries may occupy central positions. (Inkpen & Tsang 2005) examines the effect of structural, cognitive and relational dimensions of social capital on the transfer of knowledge between the network members. The study proposes that different types of networks require different kinds of facilitating conditions to transfer knowledge among network members. The size of network is also important and one-size-fits-all analyses on network may not be suitable in tackling the complexity involved in the knowledge transfer processes. Centralization of public facilities may be good for better monitoring, control and maintenance but it leads to several problems ranging from accumulation of wastes, increased load on transportation infrastructure to healthcare systems eventually deteriorating the quality of life. Therefore, it is emphasized that nena

Ravasz & Toroczkai 2010) emphasizes the need for approximation in computation of betweenness centrality as the computation of betweenness centrality is very difficult and time consuming especially for large networks with millions to billions of nodes. Such large networks may be approximated with known number of hubs and leaf nodes connected at those hubs and a general formula would help to compute the approximate centrality values of resembling network of any size and structure. Obtaining analytic expression for node centrality and the network centralization applicable to any arbitrary network is infeasible because it depends on the network topology and there are too many topologies to consider. Despite this difficulty, the analytic expressions for the network centralization corresponding to some standard network topologies should be worth exploring. Little research has been done to compare the behavior of these centrality measures in different types of network topology. Also, there is no unified measure of the level of decentralization in the network. Results on how the network centralization changes as the network gets decentralized into multiple interconnected subgroups are not known.

The network centralization based on degree, betweenness, and closeness has been defined in (Freeman 1979), but the network centralization based on eigenvector centrality is not defined in the literature. There is no literature on systematic comparison of the four important centrality measures on some standard network configurations that may help understand the interrelationships between the centrality measures and relative centralization of various network configurations. Further, there is no benchmark network centralization value available in the literature for different network configurations except for the star configuration to measure and visualize the level of decentralization in the governance or organizational structure represented in the form of a network. There is a lack of closed form solutions for benchmark network configurations except the star configuration to quantify the level of decentralization in the network structures. Therefore the centralization values obtained using small sample network may not be scalable to large network and computation of centralization values is difficult for large networks as well as it becomes difficult to approximate real world network into simpler configurations. This may give erroneous research findings about the impact of decentralization in a large networked system on its performance and dynamics. This work attempts to fill this void in the literature by providing general formulae to compute the network centralization based on all the four centrality measures for some standard types of network configurations with comparable number of nodes as in the problem under investigation to compare and visualize the level of decentralization in the network structure. These formulae are scalable to any number of nodes in the network within some constraints put to achieve structural symmetry for ease in computation. The comparative analysis of these standard network configurations with any empirical network would help in getting insights into the level of decentralization in large empirical networks.

This paper provides theoretical contribution towards analysis of the sensitivity of network centrality measures on the level of decentralization in a network. We analytically derive the formulae for the network centralization for some of the standard topologies of decentralized network and compare their centralization with the most centralized star topology. We suggest possible solutions to extend the concept of network centralization based on eigenvector centrality. Our research attempts to answer some of the following questions. How does the network centralization value based on different types on centrality measures change with the level of decentralization in the network and with the size of the network? How sensitive are these measures to the introduction of a central node to otherwise decentralized network with completely interconnected hubs? At what level of decentralization do the hub nodes become more important than the central node for any particular centrality measure? How do the centrality values scale with the size of the network? Do these centralization measures correlate with each other as the level of decentralization in the network increases? The variance in node centrality corresponding to the four centrality measures for all the types of network has also been compared.

Section 2 looks into the related research in this area. Section 3 analyses the network centralization measures based on the level of decentralization in the network. The results and discussions on the sensitivity of the centralization measures are presented in Section 4. Section 5 draws concluding remarks.

## 2. Related Research

Centrality and network centralization measures have been used extensively in social network analysis literature to study the influence of important actors and analyze network structures arising in several practical scenarios. The basic idea behind the social network analysis is that the observed network structure of actors is the outcome of actions of various actors and social institutions in the society. Opinion and behavior within a group are more homogeneous compared to that between different groups. People connected across group may have alternating ways of behaving and such people attain an advantageous brokerage position in establishing links across structural holes between distant groups in the network. People with large number of contacts (degree) often play a role of opinion leaders. Opinion leaders help in propagating information across the social boundaries between groups in a network (Burt 1999). Such people can play the role of broker in bridging different groups across the structural holes and bring novel ideas into the groups thus creating a social capital (Burt 2004). Social capital of an individual refers to the benefits he can derive through his social network. The focus of social capital research is on the features of the network that contribute to the individual, whereas with key player research the emphasis is on which individuals are important for the network (Borgatti 2006). (Friedkin 1993) has examined the relationship between the interpersonal power and interpersonal influence in resolving issues related to organizations. His findings suggests that the social structure that gives an individual social power have significant positive effect on the frequency of issue-related communications among the members of the organization that in turn have substantial effect on interpersonal influence. Two competing views on creation of social capital in a network has been presented in (Gargiulo & Benassi 2000; Burt 2001). A cohesive network provides a safety of cooperation while the structural holes provide flexibility in developing new ideas that may lead to innovations in the network. The network measures of social capital has been discussed in (Borgatti *et al.* 1998). It has been shown that the scale free network provides the fastest growth and diffusion of newly innovated knowledge (Lin & Li 2010).

(Renneboog & Zhao 2011) investigated the role of director networks on the top manager's compensation and the pay-setting process in the UK. Literature shows that both formal and informal professional and social network affects the monitoring of economic and financial activities and corporate decision making. CEOs accumulate larger social capital by setting right kind of network amongst top management and directors as different types of network enables different kinds of managerial functions and hence drives the motive for forming right kind of networks in an organization. Indirect network are built for reasons of information gathering and direct networks helps in accumulating more managerial influence. Therefore, a CEO well connected with board directors often derives significantly higher compensation. Assessing the centrality of senior managers is important for any firm in corporate decision making and pay-setting process in order to retain their competitive edge in the market. Network modeling of real world interconnected systems from diverse area is gaining attention in recent years (Uzzi 1997; Doerfel 1998; Newman 2003; Guckenheimer & Ottino 2008; Helbing 2008; Kolaczyk 2009). The topological properties of the network and identification of clusters play crucial role in understanding the internal structure and dynamics of a network (Newman 2008; Mishra *et al.* 2009). Random matrix theory and spectral methods are used to study correlation based networks and identification of clusters in a network (Edelman & Rao 2005; Kim & Jeong 2005; Newman 2006; Heimo *et al.* 2008). The concept of social network analysis has been used for analyzing interdependence structure between stock indices (Roy & Sarkar 2011a) and between the stocks (Roy & Sarkar 2011b) in the global stock market. Trading among actors in stock option market show a distinct social structural patterns that affects the direction and magnitude of price volatility (Baker 1984 ). Bounded rationality and opportunistic behavior of economic actors gives rise to restrictive micro-networks. Such restrictive micro-networks may create differentiated macro-networks in large markets leading to information asymmetry due to inefficient communication among actors.

The influence of social network on spread and cessation of smoking in a group of socially interconnected people have been investigated by (Christakis & Fowler 2008). They find that generally the whole group of smokers quit smoking together emphasizing the influence of social network on individual behavior to conform with the majority behavior in the group leading to the observed collective behavior of the entire group. Data available from online environment and information and communication systems has enabled vast amount of research in the broader domain of network science and in particular Social network analysis (Rosen *et al.* 2010). In a social network, position of actors defines the role of the actor therefore similarity among relations can be used to determine the group of actors into different classes known as equivalence classes of actors. There are several equivalence classes such as structural, isomorphic, regular etc defined based on the types of ties between the actors. The roles of individuals can in inferred from the pattern of ties that emerge due to the types of role played by the actor in the social network (Wasserman & Faust 1994). Applying modality (different classes of node types) and equivalence concepts may facilitate understanding the social processes and patterns in the complex and voluminous data that is generated through social interactions of various agents (Hanneman & Shelton 2011).

(Friedkin 1991) provides theoretical foundations for three complementary centrality measures based on elementary process model of social influence that explains why ties are formed or are broken in a social network. He classifies the three centrality measures arising due to three complementary effects namely total effect, immediate effect, and mediative effect. (Borgatti 2005) pointed out that the most commonly used centrality measures make implicit assumption about the flow processes in the network and hence the measures derived using a different assumption applied to a flow with different characteristics may lead to wrong interpretation and answers. Therefore the centrality measure used to present a particular phenomenon should match with the appropriate kinds of flow processes. The centrality values for large networks may be estimated by using a sub-sample of the network. But such estimate is likely to suffer from errors due to incomplete set of data in estimating these values. (Costenbader & Valente 2003) have investigated the impact of different levels of sampling of data from the population on the stability of centrality measures and shown that some centrality measures are more stable than others under different levels of sampling. The research highlights the impact of missing data on approximating the values of the centrality for the entire network using a smaller sample. We need a large set of data to estimate the centrality values in order to reduce the impact of missing data that may pose practical problems in many situations. Moreover, research gaps exist in analytical formulations of centrality measures in network topologies of varying degrees of decentralization levels.

## 3. Impact of Network Configuration and Decentralization on Centrality Measures

We have derived analytical expressions for all the four centrality measures for some standard network configurations with varying levels of decentralization. Such configurations arise in several practical situations (Wasserman & Faust 1994; Rivkin & Siggelkow 2007). In order to compare the sensitivity of the centrality measures, we have computed the node level centrality as well as the entire network level centralization measures based on all the four centrality measures for these network configurations. For those configurations, having complicated analytic expressions for centrality and centralization, we have computed their values numerically using MATLAB. We have also compared the variance of node centrality for these network configurations and the correlation between various centrality measures for some selected network configurations. We now present the derivations of the analytical expressions.

### 3.1 Network centralization of some network configurations

(Freeman 1979) has defined the generalized measure of graph centralization based on the differences in point centralities in the network and derived the general formula for graph centralization based on degree, betweenness and closeness centrality. The node centrality values are normalized by dividing them by the highest possible centrality value among all the nodes in the network. The general form of network centralization is given by

$$Network\ \ Centralization = \frac{\sum_{x \in U} (C_A^{\max} - C_A(x))}{Max \sum_{x \in U} (C_A^{\max} - C_A(x))} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \textbf{(1)}$$

Where is the maximum value of normalized node centrality among all the nodes based on a particular type of centrality measure, say A. The denominator denotes the maximum the most centralized configuration in case of degree, closeness and betweenness centrality. We observe some counterintuitive behavior of the denominator in case of eigenvector centrality as discussed in the following section. Variance of node centrality in the network is another measure of network centralization (Wasserman & Faust 1994). Intuitively, in a strongly centralized network the variance of node centrality from the centrality of the most central node should be the highest. The closed form expression for this measure is complicated, but the level of centralization using this metric can easily be computed numerically using the expressions derived in this paper for the normalized node centrality value for different types of equivalent nodes in the network.

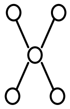
### 3.2 Network centralization based on eigenvector centrality of nodes

Eigenvector centrality is useful for analyzing the relative importance of nodes in a network such as social status of actors in a social network and detecting changes in the connectivity patterns in the neural architecture of human brain (Bonacich & Lloyd 2004; Lohmann *et al.* 2010). (Bonacich & Lloyd 2004) emphasizes that a person's social status reduces with a positive link with a notorious person and increases with a negative link with a notorious person. The network centralization based on eigenvector centrality for star configuration can be shown to be equal to $(N-1)-\sqrt{(N-1)}$. The theoretical maximum value for the expression in the denominator can be shown to be equal to N-2 and it is achieved when the network constitutes of only two connected nodes (a dyad) and N-2 independent nodes. The eigenvector centrality of a node in a network is given by the respective component of the eigenvector corresponding to the largest eigenvalue (Bonacich 1987). If we define the eigenvector centrality of a node in a network to be given by the square of respective component, the network centralization based on eigenvector centrality for star configuration can be shown to be equal to $(N-2)$. The later approach of defining the eigenvector centrality is more suitable for computing network centralization as the most central star topology achieves the highest value using this approach. The first approach does not give maximum value for star configuration used as a normalization constant in the denominator of Equation 1. For example, let's consider a network topology constructed using np+1 nodes in which one central node connected to n hub nodes, and each of the hub nodes is connected to (p-1) leaf nodes as shown in Figure 1( L). The sum of the deviation of centrality value from the maximum value in this case becomes

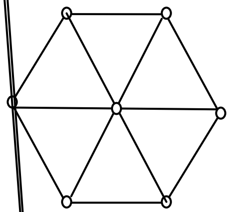
$$\left(n-\sqrt{n+p-1}\right)+(n-1)(p-1)=np-\sqrt{n+p-1}-(p-1) \text{ for } n \geq \lambda .$$

For p =2, and n>5 this expression becomes greater than the sum of deviation for star topology with same number of nodes (i.e. N = np+1) which is equal to $np-\sqrt{np}$.

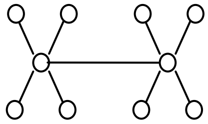
If we define the eigenvector centrality of a node in a network to be given by the square of respective component, the sum of the deviations of the node centralities from the maximum centrality value based on eigenvector centrality for a network topology in Figure 1( L) becomes



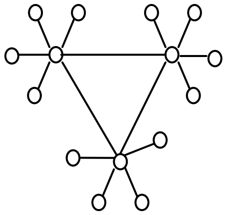
A



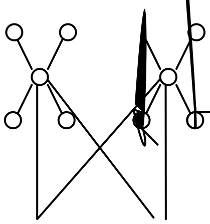
B



C



D



$(np-1)-\frac{2(p-1)}{n}$ for $n \geq \lambda$ and $(np-2n+1)$ for $n < \lambda$

| Node Type | No. of Geodesics | Normalized Centrality | Max Norm Centrality - (Norm Centrality Norm Centrality |
| --- | --- | --- | --- |

The normalization condition of the eigenvector puts the following constraints on the values of $x_1$ and $x_2$.

$$x_1^2 + (N-1)x_2^2 = 1$$

The eigenvalue equation puts the following constraints on the values of $x_1$ and $x_2$.

$$(N-1)x_2 = \lambda x_1 \qquad \text{and} \qquad x_1 = \lambda x_2$$

From these three relations the largest eigenvalue (

| Node Type | Degree | Normalized Centrality | Max Norm Centrality – Actual Norm Centrality |
|---|---|---|---|
| Hub nodes (n) | $\frac{}{n}+(n-2)$ | | |

| Node Type | Closeness Centrality | Normalized Centrality | Max Norm Centrality – Actual Norm Centrality |
|---|---|---|---|
| Hub nodes (n) | $\frac{(N-1)n}{N(2n-1)-n^2}$ | 1 | 0 |
| (N-n) Leaf nodes | $\frac{(N-1)n}{N(3n-1)-n(n+2)}$ | $\frac{N(2n-1)-n^2}{N(3n-1)-n(n+2)}$ | $\frac{n(N-2)}{N(3n-1)-n(n+2)}$ |
| Sum | | $\frac{N^2-2}{2N-3}$ | $\frac{n(N-2)(N-n)}{N(3n-1)-n(n+2)}$ |
| Network Closeness Centralization | | | $\frac{(2\quad -3)(\quad - \quad)}{(N-1)(N(3n-1)-n(n+2))}$ |

| **Node Type** | **Eigenvector Centrality** | **Normalized Centrality** | **Max Norm Centrality – Actual Norm Centrality** |
|---|---|---|---|
| Hub nodes (n) | $\frac{\lambda}{\sqrt{n\lambda^2 + (N-n)}}$ | 1 | 0 |
| (N-n) Leaf nodes | $\frac{1}{\sqrt{n\lambda^2 + (N-n)}}$ | $\frac{1}{\lambda}$ | $1-\frac{1}{\lambda}$ |
| Sum | | $n+\frac{N-n}{\lambda}$ | $\frac{(N-n)(\lambda-1)}{\lambda}$ |
| Network Eigenvector Centralization | | | $\frac{(N-n)(\lambda-1)}{\lambda((N-1)-\sqrt{N-1})}$ |

If the eigenvector centrality of a node is defined as equal to the square of the component of the principal eigenvector rather the value of the component, the value of network centralization can be shown to be given by the following equation. Both the equations give the value of network centralization for the star network as 1.

$$C_{EV} = \frac{(N-n)(\lambda^2-1)}{\lambda^2(N-2)}$$

### 3.5 Decentralized Network with a central node and n hubs

Generalizing the network centralization for star topology to a decentralized topology with one central node connected to n hub nodes where each hub node have a star like configuration with p-1 leaf nodes (i.e. excluding the central and the hub nodes)  connected to it. This network will have total np+1 nodes. We have considered both the cases viz. the number of hubs n is less than or equal to p and the number of hubs n is greater than p. The general closed form expression for variance of centrality of nodes is complicated, therefore we omit them. However, the variance of centrality for any topology can easily be computed numerically using the normalized centrality values given in the table.

**Degree Centrality:** The central node has a degree n and the hub nodes have a degree p each and the leaf node has a degree 1 each.

| Node Type | Degree | Normalized Centrality (if n ” p) | Max Norm Centrality – Actual Norm Centrality |
|---|---|---|---|
| Central Node(1) | n | $\frac{n}{p}$ | $\frac{p-n}{p}$ |
| Hub nodes (n) | $p$ | 1 | 0 |
| n(p-1) Leaf nodes | 1 | $\frac{1}{p}$ | $\frac{p-1}{p}$ |
| Sum | | 2 | $np-2n+1$ |
| Network Centralization | | | $\frac{n(p-2)+1}{(np-1)}$ |

For case n>p

| Node Type | Degree | Normalized Centrality (if n>p) | Max Norm Centrality – Actual Norm Centrality |
|---|---|---|---|
| Central Node(1) | n | 1 | 0 |
| Hub nodes (n) | $p$ | $\frac{p}{n}$ | $\frac{n-p}{n}$ |
| n(p-1) Leaf nodes | 1 | $\frac{1}{n}$ | $\frac{n-1}{n}$ |
| Sum | | 2 | $pn-2p+1$ |
| Network Centralization | | | $\frac{p(n-2)+1}{(np-1)}$ |

**Betweenness centrality:** The central node and the hub nodes fall on the geodesics between various other nodes. The leaf nodes do not lie on any of the geodesics between any other two nodes. Therefore, their betweenness is zero. The central node falls on the geodesics between any two nodes

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_2 \\ x_3 \\ x_3 \\ x_3 \\ x_3 \\ x_3 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_2 \\ x_3 \\ x_3 \\ x_3 \\ x_3 \\ x_3 \\ x_3 \end{bmatrix}$$

The normalization condition of the eigenvector puts the following constraints on the values of x1, x2 and x3.

$$x_1^2 + nx_2^2 + n(p-1)x_3^2 = 1$$

The eigenvalue equation puts the following constraints on the values of x1, x2 and x3.

$nx_2 = \lambda x_1, \quad x_1 + (p-1)x_3 = \lambda x_2, \quad x_2 = \lambda x_3$

From these four relations the largest eigenvalue ( ) and the three components x1, x2 and x3 can be computed as given below.

$$\lambda = \sqrt{n + p - 1}$$

$$x_1 = \frac{n}{\sqrt{n^2 + n\lambda^2 + n(p-1)}}, \quad x_2 = \frac{\lambda}{\sqrt{n^2 + n\lambda^2 + n(p-1)}}, \quad x_3 = \frac{1}{\sqrt{n^2 + n\lambda^2 + n(p-1)}}$$

| **Node Type** | **Eigenvector Centrality** | **Normalized Centrality** (if $\lambda \le n$ $(p-1) \le n(n-1)$) | **Max Norm centrality – Actual Norm Centrality** |
|---|---|---|---|
| Central node (1) | $\frac{n}{\sqrt{n^2 + n\lambda^2 + n(p-1)}}$ | 1 | 0 |
| Hub nodes (n) | $\frac{\lambda}{\sqrt{n^2 + n\lambda^2 + n(p-1)}}$ | $\frac{\lambda}{n}$ | $\frac{n-\lambda}{n}$ |
| n(p-1) Leaf nodes | $\frac{1}{\sqrt{n^2 + n\lambda^2 + n(p-1)}}$ | $\frac{1}{n}$ | $\frac{n-1}{n}$ |
| Sum | | 2 | ö( 1)( 1) |

For $\lambda > n$

| **Node Type** | **Eigenvector Centrality** | **Normalized Centrality** (if $\lambda > n$ $(p-1) > n(n-1)$) | **Max Norm centrality – Actual Norm Centrality** |
|---|---|---|---|
| Central node (1) | $\frac{n}{\sqrt{n^2 + n\lambda^2 + n(p-1)}}$ | $\frac{n}{\lambda}$ | $\frac{\lambda - n}{\lambda}$ |
| Hub nodes (n) | $\frac{\lambda}{\sqrt{n^2 + n\lambda^2 + n(p-1)}}$ | 1 | 0 |
| n(p-1) Leaf nodes | $\frac{1}{\sqrt{n^2 + n\lambda^2 + n(p-1)}}$ | $\frac{1}{\lambda}$ | $\frac{\lambda - 1}{\lambda}$ |
| Sum | | | $\frac{(\lambda - n) + n(p-1)(\lambda - 1)}{\lambda}$ |
| Network Eigenvector Centralization | | | $\frac{(\lambda - n) + n(p-1)(\lambda - 1)}{\lambda(np - \sqrt{np})}$ |

If the eigenvector centrality of a node is defined as the square of the corresponding component of the principal eigenvector, then the network centralization can be derived as given below.

| **Network Centralization (Eigenvector)** | (if $\lambda \le n$ i.e. $(p-1) \le n(n-1)$) | (if $\lambda > n$ i.e. $(p-1) > n(n-1)$) |
|---|---|---|
| | $\frac{(n^2 p - n) + 2(p-1)}{n(np-1)}$ | $\frac{(np - 2n + 1)}{(np-1)}$ |

| **Node Type** | **Closeness Centrality** | **Normalized Centrality** | **Max Norm Centrality – Actual Norm Centrality** |
|---|---|---|---|
| Hub node | $N-1$ | 1 | 0 |
| (N-1) Leaf nodes | $3+2(N-4)$ | $\frac{N-1}{2N-5}$ | $\frac{N-4}{2N-5}$ |
| Sum | | $\frac{N^2-4}{2N-5}$ | $\frac{(\quad 1)(\quad -4)}{2\quad -5}$ |

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_2 \\ x_2 \\ x_2 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_2 \\ x_2 \\ x_2 \\ x_2 \end{bmatrix}$$

The normalization condition of the eigenvector puts the following constraints on the values of x1 and x2.

$$x_1^2 + (N-1)x_2^2 = 1$$

The eigenvalue equation puts the following constraints on the values of x1 and x2.

$$(N-1)x_2 = \lambda x_1$$

$$x_1 + 2x_2 = \lambda x_2$$

From these three relations the largest eigenvalue ( ) and the two components x1 and x2 can be computed as given below.

$$\lambda = 1 + \sqrt{N}$$

$$\frac{\qquad}{\qquad} \qquad \sqrt{\frac{\qquad )}{\sqrt{2\sqrt{\quad}(1 \quad \sqrt{\quad})}}}$$

sequential processing department controlled by a centralized offi

$$\frac{(N-1)(N-2)}{2}-(N-1)-\frac{1}{2}.(N-3)=\frac{(N^2-6N+7)}{2}$$

and that through each of the leaf nodes (except at the two ends) is given by $\frac{1}{2}$.

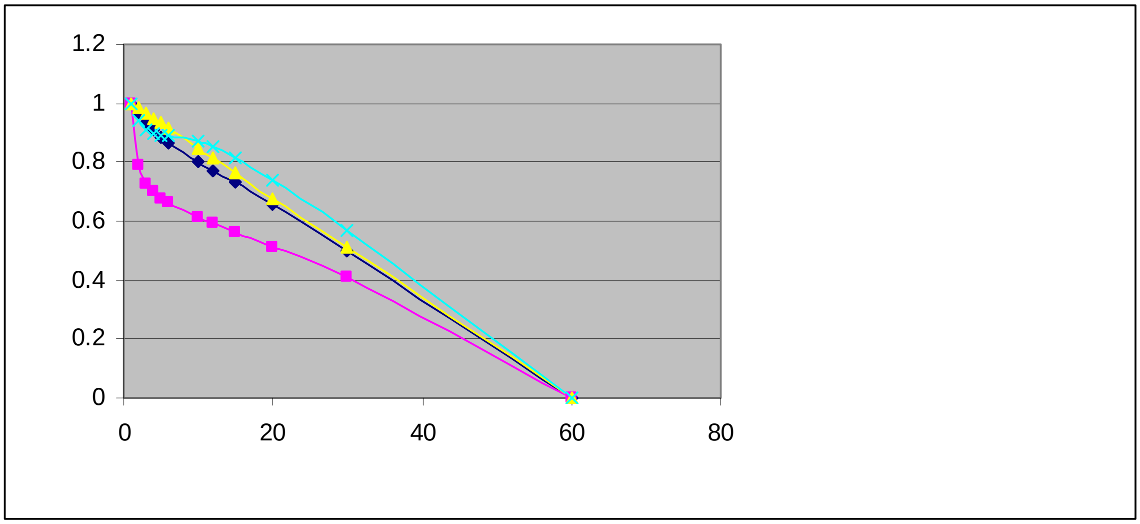
| **Node Type** | **No. of Geodesics** | **Normalized Centrality** | **Max Norm Centrality – Actual Norm Centrality** |
|---|---|---|---|
| Central node | $\frac{(N^2-6N+7)}{2}$ | 1 | 0 |
| 2 Leaf end nodes | 0 | 0 | 1 |
| (N-3) Leaf nodes | $\frac{1}{2}$ | $\frac{1}{N^2-6N+7}$ | $\frac{N^2-6N+6}{N^2-6N+7}$ |
| Sum | | $\frac{(N-4)(N-1)}{N^2-6N+7}$ | $\frac{(N-2)(N^2-5N+2)}{N^2-6N+7}$ |
| Network Centralization | | | $\frac{(2N-3)(N^2-5N+2)}{(N-1)(N^2-6N+7)}$ |

**Eigenvector centrality:** Deriving closed form expression

And $\frac{N^2-1}{4} \quad if \quad N \quad is \quad odd$

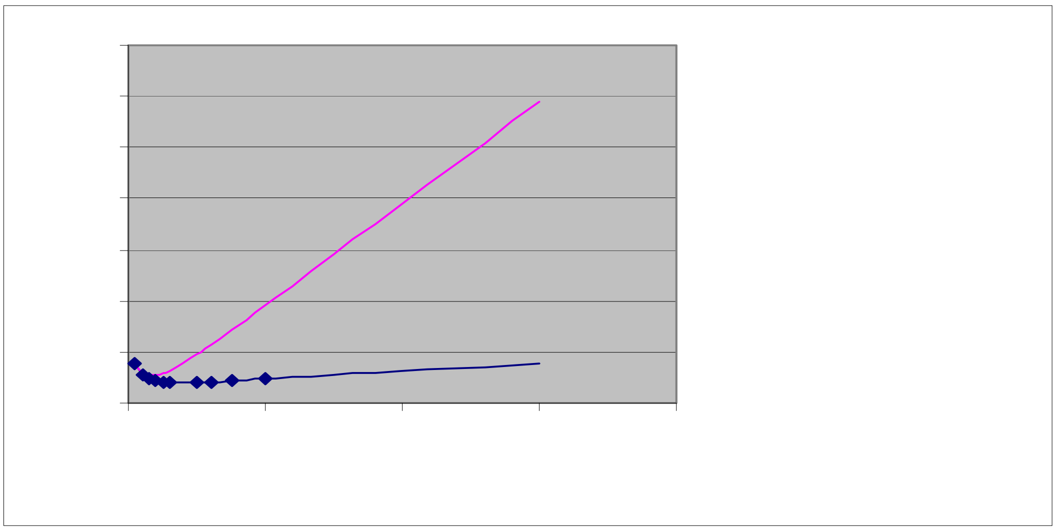
The betweenness centrality can be computed by observing that the umber of geodesics the ith node falls on = $\frac{N-3}{2}$. The largest eigenvalue for the cyclic networks can be shown to be equal to 2. The components of eigenvector representing the centrality of all the nodes can be shown to be given by $x_1 = \frac{1}{\sqrt{N}}$. Thus we can observe that all the four graph centrality of cycle graph becomes zero as all the nodes in this graph are equivalent having same relative centrality value.

**3.10 Complete graph:** In a complete graph every node is connected to all other nodes. Degree of all nodes is N-1. ClosenbeTw (nnected3145 Td(a32 90[(19 Tc eodesicss the 6 Twesicshorte c)-7 Tw 0 -01



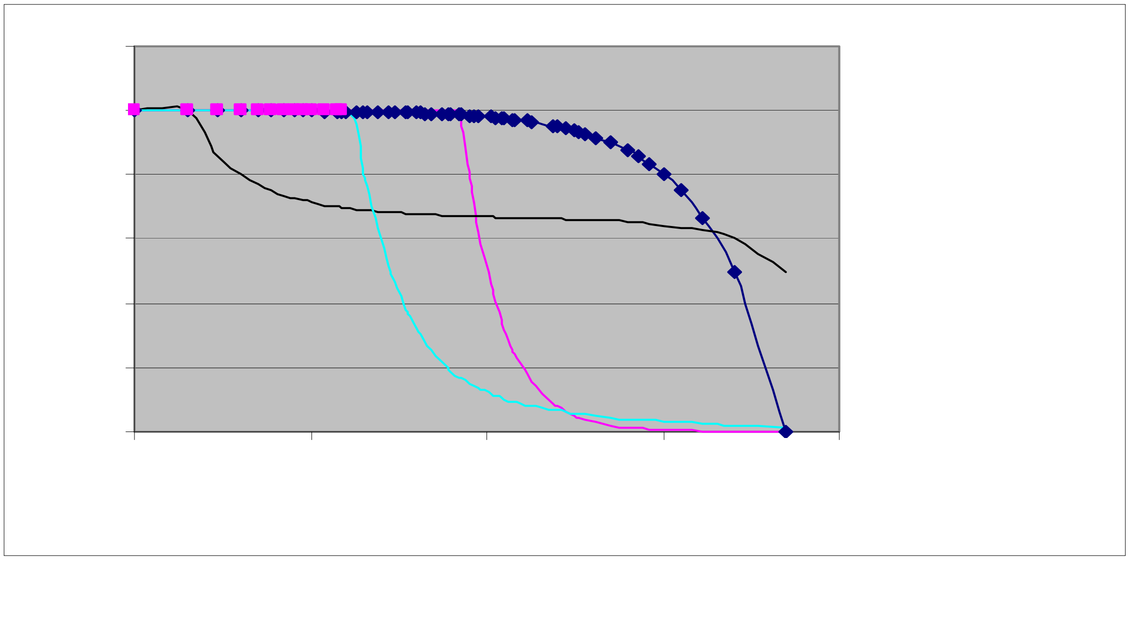
1.2
1
0.8
0.6
0.4
0.2
0
0
20
40
60
80

of nodes. Figure 4 compares the variation of the principal eigenvalue with the level of decentralization in the two types of network configurations (G and L in Figure 1) with size N equal to 60 and 61 respectively. It clearly shows that the principal eigenvalue is strongly dependent on the configuration of the network. We have shown that beyond certain level of decentralization, the principal eigenvalue of the network with a central node is less sensitive (scales as the square root of the size of the network) to the level of decentralization compared to that without a central node (scales as the size of the network). We have seen similar pattern for principal eigenvalues of the two types of network configuration with size equal to 5040 and 5041 respectively.



defining EVC in this manner won't affect the relative ranking of the nodes in the network. The network centralization based on the centrality defined as the square of the components of the principal eigenvector gives the maximum network centrality value for star configuration as shown in Figure 5.

of centrality measures as shown in Figure 7. There is a monotonous relationship between the variance of node centrality and the network centralization. We notice that the variance of all the types of node centralities increases with the level of decentralization. This measure of centralization has some counter-intuitive behaviour. For example, though cyclic and complete graphs have the least variance (zero) of node centrality, these two graphs have zero network centralization. Therefore, this cannot be used as a reliable measure of network centralization of any generic network configuration.



the betweenness and closeness centralities. An interesting finding is that the betweenness centrality is negatively correlated with all other centralities for such configuration.

**Table 1: Correlation for network with a central node and hubs**

| | Degree | Betweenness | Closeness | EVC | EVC (Comp Sqrd) |
|---|---|---|---|---|---|
| **Degree** | 1 | | | | |
| **Betweeness** | -0.37899 | 1 | | | |
| **Closeness** | 0.09161 | -0.19487 | 1 | | |
| **EVC** | 0.750984 | -0.52004 | 0.240882 | 1 | |
| **EVC (Comp Sqrd)** | 0.721989 | -0.41398 | 0.134161 | 0.96637 | 1 |

**Table 2: Correlation for network without a central node and hubs**

| | Degree | Betweenness | Closeness | EVC | EVC (Comp Sqrd) |
|---|---|---|---|---|---|
| **Degree** | 1 | | | | |
| **Betweeness** | 0.999901 | 1 | | | |
| **Closeness** | 0.911722 | 0.90862 | 1 | | |
| **EVC** | 0.996507 | 0.996831 | 0.906238 | 1 | |
| **EVC (Comp Sqrd)** | 0.999882 | 0.999995 | 0.908671 | 0.997067 | 1 |

## 5. Concluding Remarks

The sensitivity of various centrality measures on the level of decentralization of a network has been discussed. The issue with the derivation of network centralization based on eigenvector centrality of nodes of the network has been discussed and a modification in the eigenvector centrality measure for the nodes and normalization constant have been suggested in order to derive the network centralization in a unified manner compared to the other three measures. We have derived the theoretical formulae for the various network centralization measures of some standard network topologies with varying level of decentralization. Further, the [(r, the [(gactfBT/P l c and the findings in this research may be used for approximating centrality values for large networks.

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