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Continuous and strategyproof mechanisms

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Abstract

We introduce a novel notion of continuit of mechanisms, and present a complete characterization result hich sho s that: the class of VCG (Vickre [2], Clarke [3], Groves [ ]) mechanisms is the onl class of strateg proof mechanisms that satisf (eak) agent sovereignt , non-bossiness in decision, and continuit . We find that efficient mechanisms are actuall a ell-behaved subset of continuous strateg proof mechanisms.

JEL cl ssi c tion: C72; C78; D71; D 3

Keywords

Ashlagi and Serizawa [1], rules out positive transfers, and so, excludes a subset of VCG mechanisms. Further, Ashlagi and Serizawa [1] presents a characterization of decision efficiency that is independent of the one presented in this paper. They show that every strategyproof mechanism satisfying anonymity in welfare and individual rationality must be decision efficient when no positive transfers can ever be made.[3] However, in their paper, the imposition of anonymity in welfare implied that (i) the transfer functions would be independent of agent identities, and (ii) the transfer functions of each agent would be symmetric. In contrast, the present paper uses the novel continuity condition, which ensures that any well behaved strategyproof mechanism satisfying the aforementioned properties - must be decision efficient (even when positive transfers are allowed). This result allows us to completely characterize the *full* class of VCG mechanisms (including those not satisfying (i) and (ii)) *without* using decision efficiency.

To the best of our knowledge, there is no other paper that characterizes the complete class of VCG mechanisms without use of the restriction of decision efficiency.

Model

Consider an assignment problem where a single indivisible object must be allotted to any one member from the agent set

$v_{-i} := (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$, $v_{-S} := (v_i)_{i \in N \setminus S}$ and $v_S := (v_i)_{i \in S}$. Also, define for all $x \geq 0$, and all $t \in \{1, \dots, n\}$, $x^t := (x, x, x, \dots, x) \in \mathbb{R}^t_+$. Finally, for any $\varepsilon > 0$, any $t \in \mathbb{N}$, and any $y \in \mathbb{R}^t_+$; let $N_\varepsilon(y) := \{z \in \mathbb{R}^t_+ \mid \varepsilon > \|y - z\|\}$ where $\|.\|$ denotes the Euclidean norm.

We begin by defining the class of VCG mechanisms in the current setting.

Definition 1 *A mechanism* $\mu^V = (d^V, \tau^V)$ *is a VCG mechanism if and only if for all* $i \in N$*, and all* $v \in \mathbb{R}^N_+$*,*

$d_i^V(v) = 1 \Rightarrow v_i \geq v_j, \forall j \neq i.$

There exists a function $h_i : \mathbb{R}^{N \setminus \{i\}} \mapsto \mathbb{R}$ *such that*

$$\tau_i^V(v) =$$

(d) Non-bossiness in decision *For any* $i \in N$, *any* $v \in \mathbb{R}^N_+$, *and any* $x \geq 0$,

$$d_i(v) = d_i(x, v_{-i}) \Rightarrow \forall j \neq i, d_j(v) = d_j(x, v_{-i}).$$

The condition **(a)** above is a continuity condition. It requires that for all convergent sequences of profiles, if (i) some agent i gets the object at all member profiles of the sequence, and (ii) some other agent $j \neq i$ gets the object at the limit profile; then the transfers of i and j at the limit profile should make them indifferent between winning and losing the object.[6] The condition **(b)** is a boundary condition that rules out any agent getting the object by reporting a zero valuation. Note that this idea, in itself, represents a desirable property which requires no agent should get the object when she reports no desire for it. However, we impose a weaker restriction, requiring that no agent get the object by reporting zero valuation *only when there is another agent who reports a positive valuation.*

The condition **(c)** of 'weak agent sovereignty' presents the idea that each agent must *always* be able to impact the allotment process in her favour, by reporting a suitable value, should she find it preferable to do so. This restriction also been used in other mechanism design settings by Lavi, Mualem and Nisan [10] (who refer to this restriction as '*player decisiveness*'), Moulin and Shenker [15] and Marchant and Mishra [12].[7] Finally, the condition **(d)** presents a version of non-bossiness which requires that no agent be able to change any other agent's allotment decision, without changing her own decision. As argued by Thomson [25], non-bossiness of decision, in company of strategyproofness, embodies strategic restraints to collusive practices where agents form groups to misreport in a manner that changes the allotment decision to benefit one member of the group while not making any other member worse off. This condition has been used in other mechanism design settings by Nath and Sen [18] and Mishra and Quadir [13].

In this paper, we look for mechanisms in that are immune to strategic manipulation

[6] The same implication must hold for any agent who does not get the object at any member profile of the convergent sequence of profiles, but gets the object at the limit profile.

[7] Note that the restriction **(b)** would no longer be needed for our results if we use a stronger version of agent sovereignty that requires that for all i and all $v_{-i} \in \mathbb{R}^{N \setminus \{i\}}$, there exist $x^{v_{-i}}, y^{v_{-i}} \geq 0$ such that $d_i(x^{v_{-i}}, v_{-i}) \neq d_i(y^{v_{-i}}, v_{-i})$.

in reporting. In particular, we use the popular strategic axiom of strategyproofness, which eliminates any incentive to misreport on an individual level. It is defined as follows.

Definition A mechanism $\mu = (d; \tau)$ satisfies *strategyproofness* (SP) if $\forall i \in N$, $\forall v; v' \in \mathbb{R}^N_+$ such that $v_{-i} = v'_{-i}$,

$$u(d_i(v); \tau_i(v); v_i) \geq u(d_i(v'); \tau_i(v'); v_i)$$

Thus, a strategyproof mechanism guarantees that revealing the true valuation is a weakly dominant strategy for each agent in the simultaneous move game that ensues from the mechanism. The purpose of this paper is to show that the class mechanisms in that satisfy SP is same as M^{VCG}.

4 Results

We start by stating a well-known characterization of strategyproof mechanisms.

Result 1 A mechanism $\mu = (d; \tau)$ satisfies SP if and only if $\forall i \in N$ and $\forall v \in \mathbb{R}^N_+$, there exist real valued functions $K_i : \mathbb{R}^{N \setminus \{i\}}_+ \mapsto \mathbb{R}$ and $T_i : \mathbb{R}^{N \setminus \{i\}}_+ \mapsto \mathbb{R} \cup \{\infty\}$ such that

$$d_i(v) = \begin{cases} 1 & \text{if } v_i > T_i(v_{-i}) \\ 0 & \text{if } v_i < T_i(v_{-i}) \end{cases} \quad \text{and} \quad \tau_i(v) = \begin{cases} K_i(v_{-i}) - T_i(v_{-i}) & \text{if } d_i(v) = 1 \\ K_i(v_{-i}) & \text{if } d_i(v) = 0 \end{cases}$$

Proof: The result follows from Proposition 9.27 in Nisan [19] and Lemma 1 in Mukherjee [17].

Note that Result 1 allows for arbitrary tie-breaking in allocation decision of the object at any profile $v \in \mathbb{R}^N_+$ such that there exists an agent $i \in N$ with $v_i = T_i(v_j)$. In this paper, *without loss of generality*, we assume a tie-breaking rule in favour of agent 1 such that: for any profile $v \in \mathbb{R}^N_+$,

$$v_1 = T_1(v_{-1}) \Rightarrow d_1(v) = 1.$$

Thus, for any agent $i \neq 1$, this tie breaking rule does not allocate the object to i at any valuation profile where her reported value equals her threshold. Note that, our assumption of object being allocated at all profiles, in conjunction with any tie-breaking rule, requires threshold functions to be sufficiently well behaved so that whenever $v_i = T_i(v_{-i}); i \neq 1$, there exists a $j \neq i$ such that $v_j > T_j(v_{-j})$.

We begin by presenting the following theorem which plays an important role in establishing our proof. It states that for any strategyproof mechanism in Γ, the threshold functions $\{T_i(.)\}_{i \in N}$ of Result 1 must be continuous, and have finite non-negative image at all points in $\mathbb{R}_+^{n-1}$.

Theorem 1 *For any mechanism $\Gamma \in$ that satisfies S,*

1. $T_i(z) \in [0; \infty)$ *for all* $z \in \mathbb{R}_+^{N \setminus \{i\}}$, *and all* $i \in N$.
2. $\lim_{v_{-i} \to z} T(v_{-i}) = T_i(z)$ *for all* $z \in \mathbb{R}_+^{N \setminus \{i\}}$; *and all* $i \in N$:

Proof: Fix any mechanism $= (d; \,) \in$. Fix any i and any $v_{-i} \in \mathbb{R}_+^{N \setminus \{i\}}$ such that $v_{-i} \neq 0^{n-1}$. If $T_i(v_{-i}) < 0$ then, by Result 1, $d_i(0; v_{-i}) = 1$ which contradicts condition **(b)** of Definition 2. Also, if $T_i(v_{-i}) = \infty$, then $d_i(x; v_{-i}) = 0$ for all $x \geq 0$ which contradicts condition **(c)** of definition 2. Now consider the point 0^{n-1}. Note that arguing as above we can show that condition **(c)** implies $T_i(0^{n-1}) < \infty$. Consider the possibility that $T_i(0^{n-1}) < 0$ which implies that $d_i(0^n) = 1$. Now, by condition **(b)**, $d_i(0; v_{-i}) = 0$ whenever $v_{-i} >> 0^{n-1}$. And so, for any sequence of profiles $\{v^k\}$ that converges to 0^n, such that for all k, $v_i^k = 0$ and $v^k_{-i} >> 0^{n-1}$; we have $d_i(v^k) = 0$ but $d_i(0^n) = 1$, and so, condition **(a)** of Definition 2 implies that $0 \leq T_i(0^{n-1}) = 0$.

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(v)

1, $d_i(v^t) = 0$. Further, by construction, fv^tg converges to v where $v_i = T_i$

Now, we present the main result of this paper which states that the only strategyproof mechanisms in are the VCG mechanisms.

Theorem M^{VCG} *is the unique class of mechanisms in that satisfy S .*

Proof: To prove this result, we need to show that any mechanism 2 satisfies SP if and only if $= {}^{V}$. The proof of necessity follows from Result 1 and Theorem 2 above. To see the sufficiency, we simply need to show that ${}^{V}\ 2$.[8] It is easy to see that for ^{V}: (i) the object is given, at all reported profiles, to any one of the highest bidders implying that ^{V} satisfies conditions **(b)** & **(d)**; and (ii) every agent can report a value greater that all her competitors' reported values to get the object, implying than ^{V} satisfies condition **(c)**. To see that ^{V} also satisfies the continuity condition **(a)**, consider any convergent sequence of profiles $fv^t g$ with limit at $v \quad 0$, such that $d^V(v^t)$, and hence, $w(v^t)$ remains unchanged with t. Hence, we can define an agent $w\ 2\ N$ such that $w(v^t) = w$ for all $t\ 2\ \mathsf{N}$. Therefore, by definition of ^{V}, for all t, $v^t_w \quad v^t_j$ for all j

five examples of mechanisms which fail to satisfy one of these axioms, while satisfying all other properties. These examples are as follows:

- **NBD** Consider a setting where $N = \{1, 2, 3\}$, and a mechanism of the kind described in Result 1 such that: (i) for any $v \in \mathbb{R}_+^N$,

$$T_1(v_2, v_3) = v_2 + v_3;\ T_2(v_1, v_3) = \max\{v_1 - v_3, v_3\};\ T_3(v_1, v_2) = \max\{v_1 - v_2, v_2\};$$

  (ii) ties are broken in favour of agent 3, and (iii) losers receive zero transfers. Note that $d(9, 5, 3) = (1, 0, 0)$, but $d(9, 5, 4.5) = (0, 1, 0)$; implying that this mechanism does not satisfy NBD.

  It is easy to see that this mechanism satisfies C, R, SP, and WAS. Further, note that whenever $v_1 < T_1(v_{-1})$, the higher bidder $i \in \{1, 2\}$ gets the object, and so, this mechanism allocates the object at all profiles. Finally, consider sequence of profiles $(v^k) \to v$ such that for all k, (without loss of generality) $d_2(v^k) = 1$. Further, suppose that $d_2(v) = 0$. Then, by construction, $v_2^k \geq T_2(v^k_{-2})$ for all k, which implies that $v_2 = \max\{v_1 - v_3, v_3\}$, further implying that $u(1, \tau_2(v); v_2) = u(0, \tau_2(v); v_2)$.

- **C** Consider a setting where $N = \{1, 2\}$, and a mechanism of the kind described in Result 1 such that for all $v \in \mathbb{R}_+^N$:

$$T_1(v_2) = \begin{cases} 2v_2 & v_2 \leq 50 \\ v_2 & \text{otherwise} \end{cases} \quad \text{and } T_2(v_1) = \begin{cases} v_1 & v_1 < 50 \\ 50 & v_1 \in [50, 100) \\ \frac{v_1}{2} & v_1 \geq 100 \end{cases}$$

  and ties are broken in favour of agent 1 with losers receiving zero transfers. It is easy to see by Theorem 1 that this mechanism violates C. It is also easy to check that the object is allocated at all profiles, and hence, this mechanism satisfies NBD. Finally, one can easily check that this mechanism satisfies R, SP and WAS.

- Consider a setting where $N = \{1, 2\}$, and a mechanism of the kind described in

Result 1 such that for all $v \in \mathbb{R}^N_+$:

$$T_1(v_2) = v_2 - 5 \text{ and } T_1(v_2) = v_1 + 5$$

with ties broken arbitrarily and losers receiving zero transfers. It is easy to see that agent 1 is allocated an object at the profile $(0, 1)$, which violates R.

Further, it is easy to see that this mechanism allocates the object at all profiles, and hence, satisfies NBD, SP and WAS. Finally, to see that this mechanism satisfies C, consider sequence of profiles $(v^k) \to v$ such that for all k

satisfy non-bossiness in decision and agent sovereignty. These results provide new connections between continuity, strategyproofness, and efficiency in a standard mechanism design setting.

It would be difficult, but interesting, to investigate whether the presented results continue to hold for multiple identical indivisible objects, or heterogeneous indivisible objects. We leave these questions for future research.

7 Appendix

.1 Proof of Theorem 2

The proof relies on the following four lemmata.

Lemma 1 *For any mechanism* $= (d; \ ) \in$ *that satisfies S* ,

1. *For all* $v \in \mathbb{R}^N_+$ *an any* $i \in N$, $v_i > T_i(v_{-i}) \Rightarrow \{v_j < T_j(v_{-j}); \forall j \neq i\}$.

2. *For all* $v \in \mathbb{R}^N_+$ *an any* $i \in N$,

$$v_i = T_i(v_{-i}) \Rightarrow \exists j \neq i \text{ such that } v_j = T_j(v_{-j}) \text{ an } v_k \quad T_k(v_{-k}); \forall k \neq i; j \ :$$

Proof: Fix any mechanism $= (d; \ ) \in$ that satisfies SP, and any $v \in \mathbb{R}^N_+$. If there exists $i \neq j \in N$ such that $v_i > T_i(v_{-i})$ and $v_j = T_j(v_{-j})$, then by Result 1, for all $k \in N \setminus \{i; j\}$, $v_k \quad T_k(v_{-k})$. Suppose, without loss of generality, that for all $k \neq i; j$, $v_k < T_k(v_{-k})$.[9] Now, by continuity of the threshold functions (established by Theorem 1), for any $\in (0; v_i \quad T_i(v_{-i}))$, there exists $_i > 0$ such that for all $z \in \mathbb{R}^{N \setminus \{i\}}_+$ with $\|v_{-i} \quad z\| < \ _i$, $T_i(z) < T_i(v_{-i}) + \ < v_i$. Similarly, for all $k \neq i; j$, there exists $_k > 0$ such that for all $z \in \mathbb{R}^{N \setminus \{k\}}_+$ with $\|v_{-k} \quad z\| < \ _k$, $v_k < T_k(v_{-k}) \quad _k < T_k(z)$. Hence, defining $:= \min\{ \ _i; \{ \ _k\}_{k \neq i;j}\}$ (it is well defined as the number of agents is

[9] The same arguments that follow would work if there is any other agent $l \neq i; j$ such that $v_l = T_l(v_{-l})$. The only difference that would arise would be that $\tilde{\ }$ would now be defined over all agents $k \neq i; j; l$.

finite), we can infer that there exists a $\epsilon \in (0, \;)$ such that $v_i > T_i(v_j + \epsilon; v_{-\{i,j\}})$, and $v_k < T_k(v_j + \epsilon; v_{-\{j,k\}})$; $\forall k \neq i, j$. Now, since $\epsilon > 0$, by Result 1, $d_i(v_j + \epsilon; v_{-j}) = d_j(v_j + \epsilon; v_{-j}) = 1$ implying a contradiction to single indivisible object setting. Thus, condition (1) follows.

To establish condition (2) consider the possibility that there exists an $i \in N$ and $v \in \mathbb{R}^N_+$ such that $v_i = T_i(v_{-i})$, and $v_j < T_j(v_{-j})$ for all $j \neq i$. Arguing as above, there exists an $\epsilon > 0$ such that $v_j < T_j(v_i - \epsilon; v_{-\{i,j\}})$ for all $j \neq i$. By Result 1, it implies that $d_t(v_i - \epsilon; v_{-i}) = 0$ for all $t \in N$, which contradicts our supposition that the object must be allocated at all reported profiles. Hence, the condition (2) follows.

Lemma 2 *If a mechanism $\phi = (d, \tau) \in \Phi$ satisfies SP, then:*

1. *for any $v \in \mathbb{R}^N_+$ and any $i \in N$, $T_i(v_{-i})$ is non-decreasing for any change in direction of each unit vector.*[10]

2. *for any $x \geq 0$, any $i \in N$, and any $v \in \mathbb{R}^n_+$ such that $v_{-i} = x^{n-1}$ and $v_i = T_i(x^{n-1})$,*

$$v_j = T_j(v_{-j}); \forall j \neq i.$$

Proof: Fix any mechanism $\phi = (d, \tau) \in \Phi$ that satisfies SP, any $i \neq j \in N$ and any $v_{-\{i,j\}} \in \mathbb{R}^{N\setminus\{i,j\}}_+$.[11] Say there exists $0 \leq v_j^1 < v_j^2$ such that $T_i(v_j^1; v_{-\{i,j\}}) > T_i(v_j^2; v_{-\{i,j\}})$. Fix a $\epsilon \in (T_i(v_j^2; v_{-\{i,j\}}), T_i(v_j^1; v_{-\{i,j\}}))$, and consider two profiles $v, v' \in \mathbb{R}^N_+$ such that $v_i = v'_i = \epsilon$, $v_{-i} = (v_j^2; v_{-\{i,j\}})$, and $v'_{-i} = (v_j^1; v_{-\{i,j\}})$. By Result 1, $d_i(v) = 1 \Rightarrow d_j(v) = 0$, and so, $d_j(v') = 0$ as $v_j^1 < v_j^2$. But, by construction, $d_i(v') = 0$ which implies a contradiction to **(d)**. Hence, the condition (1) follows.

To establish condition (2), fix any $x \geq 0$, any profile v and any agent i such that $v_i = T_i(x^{n-1})$ and $v_{-i} = x^{n-1}$. By Lemma 1, there exists an agent $k \neq i$ such that $x =$

[10] Unit vectors are the vectors $e^1, \ldots, e^{n-1} \in \mathbb{R}^{n-1}_+$ such that each $t = 1, \ldots, n-1$ $e^t_l = \begin{cases} 1 & \text{if } t = l \\ 0 & \text{otherwise} \end{cases}$

[11] If $|N| = 2$, then the result would follow trivially from Lemma 1.

$v_k = T_k(T_i(v_{-i}); v_{-\{i;k\}}) = T_k(T_i(x^{n-1}); x^{n-2})$. Without loss of generality, suppose that $d_i(v) = 1$.[12] Now suppose there exists another agent $j \neq i;k$ such that $v_j < T_j(v_{-j})$ implying that $x = v_j < T_j(T_i(x^{n-1}); x^{n-2})$. Therefore, by **(d)** and condition (1) proved above, $v_i = T_i(x+\epsilon; x^{n-2})$ if $x + \epsilon < T_j(T_i(x^{n-1}); x^{n-2})$. Therefore, by Lemma 1 and Result 1, we get that:

$$T_i(x+\epsilon; x^{n-2}) - T_i(x^{n-1}) = \begin{cases} 0 & \text{for all } 0 \le \epsilon < T_j(T_i(x^{n-1}); x^{n-2}) - x \\ \text{positive} & \text{for all } \epsilon > T_j(T_i(x^{n-1}); x^{n-2}) - x \end{cases} \quad (1)$$

Note that by Result 1, $T_i(.)$ values must *not* depend on the value reported agent i. On the other hand, equation must hold true for all values of $x \ge 0$. Now, consider the possibility that $T_j(.)$ is independent of i's reported value. This would imply that, at any profile $\hat{v}$ where $\hat{v}_j > T_j(0^{n-2})$, $\hat{v}_i > T_i(\hat{v}_j; 0^{n-2})$, and $\hat{v}_l = 0$ for all $l \neq i;j$; the decision values $d_i(\hat{v}) = d_j(\hat{v}) = 1$, which contradicts a single object being allocated. Therefore, (1) implies that $T_i(x^{n-1})$ is a constant for all values of $x \ge 0$, and all $i \in N$. In that case, we can define n non-negative finite real numbers $K_1; K_2; \dots; K_n$ such that for any $l \in N$, $K_l = T_l(x^{n-1}); \forall x \ge 0$. Now, given the finite number of agents, we can choose a $K^* > \max_{l \in N} K_l$, and consider the profile of reports v^* where every agent i reports the same value K^*. By construction, $K_i \le T_i(v^*_{-i})$ for all i, and so, by Result 1, $d_i(v^*) = 1$ for all i which again contradicts the single object setting.

Hence, we can infer that, for all $j \neq i;k$, $v_j = T_j(T_i(x^{n-1}); x^{n-2})$ and so, the condition (2) follows.

Lemma *If a mechanism* $\psi = (d;\tau) \in \Psi$ *satisfies SP then for all* $x \ge 0$ *and all* $i \in N$,

$$T_i(x^{n-1}) = x$$

Proof: Fix any mechanism $\psi = (d;\tau) \in \Psi$ that satisfies SP. Fix any value $x \ge 0$ and any agent $i \in N$. Consider the two possibilities: (i) $T_i(x^{n-1}) < x$, and (ii) $T_i(x^{n-1}) > x$.

[12] The only other possibility is that $d_k(v) = 1$. In that case too, the same arguments would lead to the same conclusions.

Consider the possibility (i). Applying condition (1) of Lemma 1 for profile x^n, (A) $T_l(x^{n-1}) > x$ for all $l \neq i$. Now fix any $j \neq k \neq i$. Further, applying Lemma 2 for profiles $\hat{v}$ and $\bar{v}$, where $(\hat{v}_i; \hat{v}_{-i}) = (T_i(x^{n-1}); x^{n-1})$ and $(\bar{v}_k; \bar{v}_{-k}) = (T_k(x^{n-1}); x^{n-1})$, respectively; we get that (B) $x = T_j(\hat{v}_{-j}) = T_j(\bar{v}_{-j})$. Now, by Lemma 2, (**A**) and (**B**), $x = T_j(\hat{v}_{-j}) \quad T_j(x^{n-1}) \quad T_j(\bar{v}_{-j}) = x$, which establishes that $T_j(x^{n-1}) = x$, which contradicts (**A**). Hence, possibility (i) cannot hold.

For possibility (ii), consider the profile x^n, and note that, by Result 1, $d_i(x^n) = 0$. So, there exists a $j \neq i$ such that $d_j(x^n) = 1$. Now, if $x > T_j(x^{n-1})$, then arguing as above, we can show that there exists some $l \neq j$ such that $x = T_l(x^{n-1})$, which would contradict Lemma 1. Now if $x = T_j(x^{n-1})$, then by applying Lemma 2 to the profile x^n, we get that $x = T_i(x^{n-1})$, which contradicts the possibility (ii). Hence, the result follows.

Lemma 4 *If a mechanism* $= (d; \;) \in$ *satisfies S* *then for all* $i \in N$*, an* *for all* $v \in \mathbb{R}^N_+$*,*

$$T_i(v_{-i}) = \max_{j \neq i} v_j$$

Proof: Fix any mechanism $= (d; \;) \in$ that satisfies SP. Also, fix any agent $i \in N$, and any $z \in \mathbb{R}^{n-1}$. Without loss of generality, assume that $z = (z^1; z^2; \ldots; z^{n-1})$ where $z^k \quad z^{k+1}$ for all $k = 1; \ldots; n-2$. Therefore, we need to show that $T_i(z) = z^1$. For the sake of notational simplicity, let $:= z^1$.

Now, fix any > 0 and consider the profiles v and v such that $v_i = + ; v_{-i} = {}^{n-1}$ and $v_i = \quad ; v_{-i} = {}^{n-1}$. By Lemma 3, $T_i({}^{n-1}) =$, and so, by construction, $v_i > T_i({}^{n-1})$ and $v_i < T_i({}^{n-1})$. Now, by condition (1) of Lemma 2 and construction of , $T_i({}^{n-1}) \quad T_i({}^{n-2}; z_n) \quad \ldots \quad T_i(z)$ implying that $v_i > T_i(z)$. Arguing similarly for profile v , we get that $v_i < T_i(z)$. Thus, we get that for all > 0,

$$v_i < T_i(z) < v_i$$

which implies that $T_i(z) = \; = z^1$. Hence the result follows.

It easy to see that the threshold function specified in Lemma 4 requires the object to be allotted to the highest bidders at all valuation profiles, and hence, describes an efficient mechanism.

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