Characterization of Vickrey auction with reserve price for multiple objects

Conan Mukherjee *
Assistant Professor
Economics Group
Indian Institute of Management Calcutta
D. H.

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Ranojoy Basu [1] and Conan Mukherjee [y2]

[1]Department of Economics and Finance, Indian Institute of Management Udaipur, India

[2] Economics Group, Indian Institute of Management Calcutta, India

Abstract

This paper completely characterizes Vickrey auction with reserve price [VARP], in single and multiple objects settings, using normative and strategic axioms. In particular, it provides a topological interpretation of reserve price as the in mum of a particular set of non-negative real numbers.

In the single object case, we nd that a strategyproof mechanism satis es anonymity in welfare, agent sovereignty and non-bossiness in decision if and only if it has a VARP allocation rule. To extend this result to the multiple objects setting, we introduce a continuity condition and show that any continuous and strategyproof mechanism satis es the aforementioned properties (and a mild regularity condition) if and only if it has a VARP allocation rule.

JEL classi cation : C72; C78; D71; D63

Keywords: Anonymity in welfare, agent sovereignty, non-bossiness in decision, continuity, strategyproof mechanism

1 Introduction

It is well known that reserve pricing at auctions is an important method of ensuring that the seller revenue is not too low (Ausubel and Cramton [3]). Vickrey auctions, on other hand, ensure that the objects are allocated e ciently and that agents have no incentive to misreport irrespective of what other agents are reporting. Therefore, Vickrey auction with reserve price [VARP][1] is a useful mechanism for accomplishing both objectives of e cient allocation of objects and avoidance of low seller revenues. It is, therefore, no

Email: ranojoy.basu@iimu.ac.in

[y]Email: conanmukherjee@gmail.com

[1]Vickrey auction with reserve price is a mechanism with a special allocation rule where objects are allocated only to agents whose bids are not less than the reserve price. Further, winners of object pay the maximum of the reserve price and the greatest losing bid as price and non-winners pay nothing.

surprise that auctioneers have been recorded to be using VARP as early as 1897.[2] While revenue generation properties of VARP have been well documented over time, there is a dearth of literature on ethical properties of VARP. This is in contrast to a large literature providing normative characterizations of Vickrey auction without reserve prices.[3] This paper attempts to ll this gap by completely characterizing VARP, both in single and multiple objects settings (with unit demand), using normative axioms.

We present the idea of ethical mechanisms by invoking two popular notions of fairness: *anonymity in welfare* and *agent sovereignty*.[4] A mechanism is said to satisfy anonymity in welfare if utility levels of any two agents get interchanged, when their valuations are interchanged with all other agents' valuations remaining unchanged. A mechanism is said to satisfy agent sovereignty if it provides each agent with some opportunity to get an object, irrespective of what the other agents are reporting.

Further, we describe the idea of mechanisms being immune to manipulation by invoking the concept of *strategyproofness*. A mechanism is said to be strategyproof if truth-telling is a weakly dominant strategy for all agents in the direct revelation game induced by it.

We use an additional axiom of non-bossiness which requires that no agent be able to a ect the allocation decision of another agent without a ecting her own allocation decision. Since this is a di erent version of the conventional non-bossiness axiom of Satterthwaite and Sonnenschein [26], we call it *non-bossiness in decision*.[5] As argued by Thomson [29], non-bossiness of decision, in company of strategyproofness, embodies strategic restrictions that discourage collusive practices where agents form groups to misreport in a manner that changes the allotment decision to bene t one member of the group while not making any other member worse o .

In the single object case, we show that a strategyproof mechanism satis es anonymity, agent sovereignty and non-bossiness in decision only if it has an allocation rule same as that of a VARP.[6] Then we completely characterize the class of mechanisms that satisfy anonymity, agent sovereignty, strategyproofness and non-bossiness in decision. Any mechanism in this class satisfies a mild *zero-utility* condition (requiring that any agent with zero valuation for the object should get zero utility by participating in the mechanism), if and only if it is a VARP.

Unfortunately, these characterizations fail to hold in the multiple homogeneous objects case straightaway. That is because with multiple objects, any number of objects may be withheld by the planner leading to a proliferation of the number of possible decisions at any valuation profile. For example, when there are three objects to be allocated; at any valuation profile, the planner must choose from four possible decisions of allocating $k \in \{0, 1, 2, 3\}$ objects. In contrast, with a single object to allocate; at any valuation profile, the planner has only two possible choices of either allocating the object or not. To address the subsequent technical complexities, we introduce a continuity condition, and show that any *continuous* mechanism satisfies anonymity, agent sovereignty, non-bossiness in decision, strategyproofness and zero-utility (and a mild regularity condition); if and only if it is a VARP. Thus, our paper completely characterizes the class of VARP in both single object and *multiple* objects settings.

1.1 Relation to literature

Perhaps the most popular paper on reserve pricing is Myerson [19]. Myerson [19], in an independent private value setting for a single indivisible object, identifies a particular VARP as *one* of the (Bayes-Nash incentive compatible) revenue maximizing mechanisms under the assumptions of: (i) symmetric bidders, (ii) distribution of valuations satisfying a regularity condition and (iii) the planner knowing this distribution with certainty. Further, Myerson [19] obtains a revenue maximizing mechanism involving different reserve prices for different agents if assumption (i) is violated. In contrast, for the single object case, our paper uses the same independent private value setting, without making the assumption (i) or any other distributional assumption, to show that any mechanism is an ethical (anonymous, agent sovereign and non-bossy) and strategyproof mechanism, if and only if it is a VARP. Thus, our result provides an interpretation of VARP (and hence, use of single *identical* reserve price across all bidders) even when bidder valuations are not symmetrically distributed. Additionally, unlike any other paper that we are aware of, our paper presents a characterization of VARP for multiple objects.

Some other papers, particularly relevant to our analysis are, Mishra and Quadir [14], Sakai [23], Klaus and Nichifor [10], and Tierney [30]. Mishra and Quadir [14] focus only on the single object allocation problem with money, and characterize the class of strategyproof and non-bossy (in decision) allocation rules. They show that for any reported valuation: the utility vector generated by the chosen allocation must be consistent to maximization of some monotone an implication of failing to meet the single reserve price for the real objects[9]. Further, the mechanisms characterized by Tierney [30], when reduced to single object setting entail a separate (possibly positive) reserve price for getting no object, which is contrary to our ndings. Hence, our results are of independent interest to theirs. Finally, instead of treating reserve prices as a parameter, we present a topological interpretation of reserve price where it gets endogenously determined as an in mum of a special set of real numbers that follow from our axioms.

From a purely strategic perspective (without any normative axiomatic structure), a few notable recent works on reserve prices and their welfare and revenue e ects are: Hu, Matthews and Zu [8], Kotowski [11], and Sano [25]. Unlike our paper, all these papers adopt the strategic perspective of Bayes' Nash incentive compatibility, under some chosen prior distribution of private informations.

The paper proceeds as follows. Section 2 presents the model and de nitions. Section 3 presents the results on single and multiple objects. Section 4 discusses the independence of axio-253utivxio9

As mentioned earlier, NBD embodies a strategic barrier to collusive practices where agents form groups to misreport in a manner that changes the allotment decision to benefit any one member of the group while not making any other member worse off[11].

The following two definitions pertain to two different notions of fairness. They describe ethically desirable behaviour that a mechanism should exhibit in an idealized state of nature where there is no private information (that is, planner knows every agent's true valuation). The first definition states the fairness concept of anonymity in welfare which requires that utility derived from an allocation by any agent be independent of her identity. The second definition states the fairness idea that each agent should have an opportunity to get an object, irrespective of what the other agents are reporting[12].

Definition 4. A mechanism $(d^m; \ ^m)$ satisfies *anonymity in welfare* (AN) if for all $i \in N$, all $v \in R_+^N$ and all bijections $\ : N \mapsto N$,

$$u(d_i(v); \ _i(v); v_i) = u(d_{\ i}(\ v); \ _{\ i}(\ v); v_{\ i})$$

where $\ v := \left(v_{\ ^{-1}(k)}\right)_{k=1}^{n}$.

Definition 5. A mechanism $(d^m; \ ^m)$ satisfies *agent sovereignty* (AS) if for all $i \in N$ and all $v \in R_+^N$, there exists $v_i^0 \in R_+$ such that

$$d_i^m(v_i^0; v_{-i}) = 1$$

Finally, the following axiom implies the fairness perception that if an agent has zero valuation for the object, then the agent must not get a positive or negative utility by merely participating in the mechanism.

Definition 6. A mechanism $(d^m; \ ^m)$ satisfies *zero-utility* if for all $i \in N$ and all $v_{-i} \in R_+^{N \setminus \{i\}}$,

$$u(d_i^m(0; v_{-i}); \ _i^m(0; v_{-i}); 0) = 0:$$

Note that for our single object setting, this zero-utility condition is logically equivalent to the non-imposition condition of Sakai [23][13].

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[11] See Thomson [29]. Also to see the kind of undesirable mechanisms that NBD excludes, consider the following example. For any profile v: (i) if there exists an agent i such that $v_i \in [0; b(1))$ and v_i is [illegible]

3 Main results

For the sake of simplicity of notation, henceforth, we suppress the superscript m while describing a mechanism $(d^m; {}^m)$ whenever the number of objects being allocated is clear from the ambient context.

We begin by noting the following well known result which establishes that the decision rule implicit in any strategyproof mechanism must be non-decreasing in one's own reported value.[14] In particular, for any agent i and any profile of valuations v_{-i}, there must exist a threshold price $T_i(v_{-i})$ such that: i gets an object if v_i strictly exceeds $T_i(v_{-i})$ and fails to get an object if v_i is strictly less than $T_i(v_{-i})$. Further, if a strategyproof mechanism satisfies AS, then these threshold prices must be finite. Finally, SP and AS imply that the transfer of agent i when getting the object, must exceed that when not getting the object, by $T_i(v_{-i})$.[15]

Fact 1. Any mechanism $(d; \ )$ satisfies SP and AS, if and only if $\forall i \in N$ and $\forall v_{-i} \in \mathbb{R}_+^{N \setminus \{i\}}$, there exist real valued functions $K_i : \mathbb{R}_+^{N \setminus \{i\}} \mapsto \mathbb{R}$ and $T_i : \mathbb{R}_+^{N \setminus \{i\}} \mapsto \mathbb{R}$ such that

$d_i(v$TR

Theorem 1. A mechanism $(d; \ )$ satisfies properties AN, AS, NBD and SP only if $\exists\, r \geq 0$ such that for all $i \in N$ and all $v \in R^N_+$,

$$d_i(v) = \begin{cases} 1 & \text{if } v_i > \max\{v_{-i}(1); r\} \\ 0 & \text{if } v_i < \max\{v_{-i}(1); r\} \end{cases}$$

Proof: We accomplish this proof in three stages[17]. First, in Lemma 2 of Appendix, we establish existence of a real which is well defined with respect to a set of valuations where at least one object is allocated. Then, in subsection 7.2 of Appendix, we show that for all v and all i, (i) $v_i < \max\{v_{-i}(1); \ \}$ implies $d_i(v) = 0$, and (ii) $v_i > \max\{v_{-i}(1); \ \}$ implies that $d_i(v) = 1$. This allows us to establish existence of a reserve price $r :=$ such that $T_i(v_{-i}) = \max\{v_{-i}(1); r\}$ for all v and all i. □

Remark 2. Kazamura, Mishra and Serizawa [9], henceforth, referred to as KMS, show that any mechanism satisfying AN, SP and `loser payment independence' (requiring that loser at any profile pay the same amount irrespective of her preference for the object), must be an adjusted Vickrey auction with a variable reserve price. Theorem 1 complements this result by showing that: in a quasilinear setting, any mechanism satisfying AN, AS, SP and NBD, must have an allocation rule same as that of a VARP (that is, uses a common reserve price).

Note that, for the single object case, our tie breaking rule implies that for any v with $v_i \ \ T_i(v_{-i}); \forall i \in N$, the object is allocated to the top most agent in $Y(v)$ according to the order $1 \ \ 2 \ \ \ldots \ \ n$. Therefore, Theorem 1 provides a novel topological interpretation to the reserve price value of a VARP. That is, it establishes that the reserve price used in a VARP mechanism, must also be the infimum of a set S consisting of non-negative real numbers satisfying the following property: if all agents bid the same number from S, then at least one object is allocated. As we shall see later, this interpretation continues to hold (in Proposition 3) when there are more than one objects to allocate. This idea is expressed in the following corollary.

Corollary 1. For any mechanism $(d^{1r}; \ ^{1r}) \in \ ^1$,

$$r = \inf\{x \geq 0 : x^n \in B_0^1\}$$

Proof: It is easy to that any VARP satisfies AN, AS, NBD, and SP. Hence, from proof of Theorem 1, the result follows. □

Next, we define a special class of mechanisms that employ uniform reserve prices in their allocation and transfer rules.

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[17] See Appendix for full details.

Definition 7. Let M^1 be the class of mechanisms (d, τ) such that for all $i \in N$ and all $v \in \mathbb{R}^N_+$,

$$d_i(v) = \begin{cases} 1 & \text{if } v_i > \max\{v_{-i}(1), r\} \\ 0 & \text{if } v_i < \max\{v_{-i}(1), r\} \end{cases}$$

$$\tau_i(v) = \begin{cases} K(v_{-i}) - \max\{v_{-i}(1), r\} & \text{if } d_i^r(v) = 1 \\ K(v_{-i}) & \text{if } d_i^r(v) = 0 \end{cases}$$

where $K : \mathbb{R}^{n-1}_+ \mapsto \mathbb{R}$ is a symmetric function.[18]

Thus, M^1 is a special class of mechanisms with the VARP allocation rule. It contains an interesting sub-class of mechanisms with this allocation rule but not the VARP transfer rule. This is the class of *maxmed* mechanisms introduced by Sprumont [27]. These mechanisms belong to M^1 and can be obtained by setting

$$K(v_{-i}) = \text{med}\left(0, v_{-i}(1) - r, \frac{r}{n-1}\right); \forall v \in \mathbb{R}^n_+; \forall i \in N;$$

where for any three real numbers x, y, z, $\text{med}\{x, y, z\}$ denotes the median on the three numbers.

The following theorem completely characterizes M^1.

Theorem 2. Any mechanism (d, τ) satisfies AN, AS, NBD and SP if and only if $(d, \tau) \in M^1$.

Proof: See Appendix. □

Note that all mechanisms in M^1 which have a special $K(.)$ function such that $K(z) = 0$ for all $z \in \mathbb{R}^{n-1}_+$, are VARP. That is, the class of VARP mechanisms for single object, Γ^1 is a subset of M^1, i.e., $\Gamma^1 \subset M^1$. We use this relation to obtain the following corollary which completely characterizes Γ.

Corollary 2. A mechanism (d, τ) satisfies AN, AS, NBD, SP and zero-utility if and only if $(d, \tau) \in \Gamma^1$.

Proof: The proof of sufficiency is easy to check. To see the necessity, fix any $i \in N$ and any $v_{-i} \in \mathbb{R}^{n-1}_+$. Consider the profile $(0, v_{-i})$. From Theorem 1 and zero-utility condition, it follows that $u_i(d_i(0, v_{-i}), \tau_i(0, v_{-i}); 0) = -K(v_{-i}) = 0$. Hence, the result follows. □

Remark 3. Corollary 2 also follows from KMS. As mentioned earlier, they show in this discussion paper that in a single object setting with general (possibly non-quasilinear)

[18] A function of $k \in N$ variables is said to be *symmetric* if the function value at any k-tuple of arguments is the same as the function value at any permutation of that k-tuple.

preferences, any mechanism satisfies AN, SP and loser payment independence, if and only if it is an adjusted Vickrey auction with a variable reserve price. In our setting: (i) SP and zero-utility condition imply the KMS axiom of loser payment independence, (ii) NBD rules out Vickrey mechanisms where the reserve price may depend on preference of other agents. Thus, their result implies our Corollary 2. Finally, dropping AS from the statement of Corollary 2 would lead to an additional trivial mechanism that never gives out the object and charges zero transfers.[19]

3.2 Multiple homogeneous objects: $m > 1$

In this section we study the case where number of objects/copies m can take any integer value from 2 to $n-1$. Ideally, the results in single object case should translate directly to the multiple homogeneous objects (with unit demand) setting. However, that is not the case. The reason for this are the following two complications that arise out of the multiple objects setting.

The first complication is that, at any profile, it no longer follows from any one agent getting an object, that other agents get no objects. Thus, the inherent externality of the single object setting, becomes very weak when $m > 1$. The second complication is other hand, the regularity of a mechanism requires that at any valuation profile: if no objects are allocated when $m \geq 2$ copies are available, then no objects must be allocated when $m - 1$ copies are available. This property rules out strange mechanisms where abundance of objects leads to scarcity in allocations.[21]

The following theorem states that any continuous regular mechanism satisfying AN, AS, NBD and SP, must have an allocation rule same as that of a VARP (for m objects). That is, any such mechanism must have an associated reserve price that is common across all agents.

Theorem 3. *A continuous regular mechanism $(d; t)$ satisfies properties AN, AS, NBD and SP only if $\exists\, r \geq 0$ such that for all $i \in N$ and all $v \in \mathbb{R}_+^N$,*

$$d_i(v) = \begin{cases} 1 & \text{if } v_i > \max\{v_{-i}(m); r\} \\ 0 & \text{if } v_i < \max\{v_{-i}(m); r\} \end{cases}$$

Proof:

Proof: It is easy to check that any VARP mechanism is continuous and satisfies AN, AS, NBD, SP. Hence, from proof of Theorem 3, the result follows. □

Now, as in the single object case, we define special class of mechanisms M^m, which employ reserve prices in their allocation of $m > 1$ objects and corresponding transfer rules.

Definition 10. Let M^m be the class of mechanisms $(d; \ )$ such that for all $i \in N$ and all $v \in R^N_+$,

$$d_i(v) = \begin{cases} 1 & \text{if } v_i > \max\{v_{\ i}(m); r\} \\ 0 & \text{if } v_i < \max\{v_{\ i}(m); r\} \end{cases}$$

$$_i(v) = \begin{cases} K(v_{\ i}) \quad \max\{v_{\ i}(m); r\} & \text{if } d_i(v) = 1 \\ K(v_{\ i}) & \text{if } d_i(v) = 0 \end{cases}$$

where $K : R^{n\ 1}_+ \mapsto R$ is a symmetric function.

Similar to the single object case, we present the following proposition, which completely characterizes M^m.

Theorem 4. Any continuous regular mechanism $(d; \ )$ satisfies AN, AS, NBD and SP if and only if $(d; \ ) \in M^m$.

Proof: See Appendix. □

Again as observed in the single object case, all mechanisms in M^m which have a special $K($

4.1 Corollary 2

This theorem uses the axioms of AN, AS, NBD, SP and zero-utility to characterize the class of VARP mechanism in the single object case. To show independence of axioms, agents $\{1;2\}$. Suppose that for all $x \geq 0$,

$$K_1(x) = -x;\; K_2(x) = \max\{0; x - \epsilon\};\; \text{and } T_1(x) = -x + \epsilon;\; T_2(x) = \max\{0; x - \epsilon\}$$

where $\epsilon > 0$. Recall that, as argued in section 5, $(d;t)$ satisfies NBD, SP, AS and zero-utility. By Proposition 1, $(d;t)$ does not satisfy AN as the $T_i(.)$ functions depend on agent identities.

(v) NBD Consider a mechanism $(d;t)$ belonging to the class described by Fact 1 such that for all $i \in N$ and $z \in \mathbb{R}_+^{n-1}$

$$K_i(z) = 0 \text{ and } T_i(z) = \begin{cases} \max\{z(1); 20\} & \text{if } z(1) > 10 \\ z(1) & \text{if } z(1) \leq 10 \end{cases}$$

Note that by Fact 1, $(d;t)$ satisfies SP and AS. Also, it can be easily seen that it satisfies zero-utility. Further, at any valuation profile, if valuations of any pair of agents are changed, their utilities get interchanged. Hence $(d;t)$ satisfies AN. To see that this mechanism violates NBD, consider a three agent setting where the valuation profile $(v_1;v_2;v_3) = (15;8;7)$. Note that according our decision rule $d(15;8;7) = (1;0;0)$. But if agent 2 unilaterally changes her reported valuation to 11, the decision changes to $d(15;11;7) = (0;0;0)$, where agent 2 continues to not

information. Now, x any 2 f 0; 1g, any agent i 2 N , and consider any sequence f v^n g such that (i) v^n 0, (ii) converges to some v^0 0, and (iii) for all n, $d_i(v^n)$ = .
Now if = 1 and d_i(v

that for all i 2 N, z 2 $R_+^{n\ \ 1}$,

$$K_i(z) = 0 \text{ and } T_i(z) = \ z(1) + \ z(n \quad 1)$$

for all $i \in N$, $z \in R_+^{n-1}$,

$$K_i(z) = 0 \text{ and } T_i(z) = \begin{cases} 10 & \text{if } z(1) \in [0 \end{cases}$$

x. Hence, $(d^1; \ ^1)$ is a mechanism, which does not employ a VARP allocation rule, and satis es AS, ETE, NBD, SP - but not AN. Thus, our characterization of VARP allocation rule crucially depends on stronger implications of AN.

6 Conclusion

This paper provides a justi cation to reserve pricing at auctions using normative and strategic axioms unrelated to revenue considerations. In particular, it provides a topological interpretation of a reserve price as the in mum of the set of non-negative real numbers satisfying the following property: if all agents bid the same number from this set, then at least one object is allocated. Finally, it provides complete characterizations of VARP in single and multiple objects settings. Whether these results continue to hold

Suppose there exists some $z \in R^{n-1}_{+}$ such that $T_1(z) \neq T_2(z)$. W. l. o. g. suppose that $T_1(z) > T_2(z)$. Construct the profile v such that $v_{-1} = z$ and $v_1 \in (T_2(z); T_1(z))$. Then, from Fact 1 it follows that $d_1(v) = 0$ since $v_1 < T_1(v_{-1}) = T_1(z)$ by construction. Now, consider the profile $v' = (v'_1; v'_2; v_{-12})$ where $v'_1 = v_2$ and $v'_2 = v_1$. Note that $v'_{-2} = v_{-1} = z$. Therefore, $d_2(v'$

Lemma 2. A mechanism $(d; \;)$ satisfies NBD and SP only if $\exists \;\; 0$ such that $\forall\, x \;\; 0$,

$$x < \;\; \Longrightarrow \;\; x^n \in B_0^m \quad \text{and} \quad x > \;\; \Longrightarrow \;\; x^n \notin B_0^m$$

Proof: Fix any mechanism $(d; \;)$ satisfying NBD and SP. Suppose that there exist $0 \;\; x < y$ such that $x^n \notin B_0^m$ and $y^n \in B_0^m$. W. l. o. g. suppose that $d_i(x^n) = 1$ for all $i = 1; \ldots; l$ where $l \in \{1; \ldots; m\}$ (that is, l objects are allocated at profile x^n). Define the sequence of profiles $(p^k)_{k=1}^{l}$ where $p^1 = (\;y; x^n{}_{1})$ and for all $2 \;\; k \;\; l$, $p^k = (\;y; p^k{}_{k}{}^{1})$. By NBD and SP, for all $1 \;\; i \;\; l$, $d_i(x^n) = 1 \Longrightarrow d_i(p^1) = 1 \Longrightarrow d_i(p^2) = 1 \Longrightarrow \ldots \Longrightarrow d_i(p^l) = 1$ and so, $p^l \notin B_0^m$. Similarly construct another sequence of profiles $(q^k)_{k=l+1}^{n}$ such that $q^{l+1} = (\;x; y^n_{\{\;l+1\}})$ and for all $l+2 \;\; k \;\; n$, $q^k = (\;x; q^k{}_{k}{}^{1})$. By SP and NBD, $y^n \in B_0^m \Longrightarrow q^{l+1} \in B_0^m \Longrightarrow q^{l+2} \in B_0^m \Longrightarrow \ldots \Longrightarrow q^n \in B_0^m$. By construction, $q^n = p^l$ and hence, contradiction. Therefore, for any $x \;\; 0$, if $x^n \notin B_0^m$ and then $\forall\, y > x$ it must be that $y^n \notin B_0^m$. Thus, if the set $\{x \;\; 0 : x^n \notin B_0^m\}$ is non-empty, then the result follows from the choice of $\; := \inf\{x \;\; 0 : x^n \notin B_0^m\}$. If $\{x \;\; 0 : x^n \notin B_0^m\} = \emptyset$ then no objects are allocated at any profile where all agents have bid the same value. In this case the result follows by assigning $:= \infty$. □

The following lemma shows that if $\; > 0$ then no object is allocated at any profile where the highest valuation is strictly less than .

Lemma 3. A mechanism $(d; \;)$ satisfies AN, NBD and SP only if $\forall\, v \in [0; \;)^n$, $v \in B_0^m$.

Step 2

In this step, we establish that $v_i > \max\{v_{-i}(1); \ \} \Rightarrow \ d_i(v) = 1$, for all v and i.

To see this, fix any $i \in N$ and any profile $v \in \mathbb{R}^n_+$ such that $v_i > \max\{v_{-i}(1); \ \}$. Note that, either $v_i = v(1) > v(2) >$ or $v_i = v(1) > \ \ v(2)$. We analyze each of the two cases below, and show that in each case $d_i(v) = 1$

Case 1: $v_i = v(1) > v(2) >$

By Lemma 2, $\overline{v(2)}^n \notin B^1_0$ and so, from Remark 1 and Proposition 1, it follows that $T(\overline{v(2)}^{n-1}) = v(2)$. Construct a sequence of profiles $\{p^k\}^n_{k=1}$ such that $p^1 = \overline{v(2)}^n$, $p^2 = (v_i; p^1_{-i})$ and $\forall\, 3 \leq k \leq n$, $p^k = (v_{t_k}; p^{k-1}_{-t_k})$ where $t_k \in \{j \in N \mid v_j = v(k)\}$ (by the tie-breaking rule, this set is a singleton set). Further, $T(p^1_{-i}) = T(\overline{v(2)}^{n-1}) = v(2)$ and so under the supposition $v_i = v(1) > v(2)$, $d_i(p^2) = 1$. Since $m = 1$ it follows that $d_j(p^2) = 0; \forall j \neq i$. Moreover, by SP and NBD, for all $j \in N$, $d_j(p^2) = d_j(p^3) = \dots = d_j(p^n)$. Since, by construction, $p^n = v$, we get that $d_i(v) = 1$ and $d_j(v) = 0$ for all $j \neq i$.

Case 2: $v_i = v(1) > \ \ v(2)$

Consider the sequence of profiles $\{p^k\}^n_{k=0}$ where $p^0 = \overline{\ \ +\ \ }^n$ and $\in (0; v_i \ \ )$. For all $1 \leq k \leq n$, $p^k = (v_{t_k}; p^{k-1}_{-t_k})$ where $t_k \in \{j \in N \mid v_j = v(k)\}$ (as mentioned before, this set is a singleton set by the tie-breaking rule). Together with Remark 1, Lemma 2 and Proposition 1, we get that $p^0 \notin B^1_0$ which implies that $T(p^0_{-j}) = \ \ + \ \ $ for all $j \in N$. Further, by construction, $p^1_i = v_i$ and $p^0_{-i} = p^1_{-i}$. Therefore, $p^1_i > T(p^1_{-i}) = \ \ + \ \ $ and so, from Fact 1 it follows that $d_i(p^1) = 1$. Since $m = 1$, we can then claim that $d_j(p^1) = 0$ for all $j \neq i$. Hence, by SP and NBD, for all $j \in N$, $d_j(p^1) = d_j(p^2) = \dots = d_j(p^n)$. By construction, $p^n = v$ which implies that $d_i(v) = 1$. □

7.3 Proof of Theorem 2

The sufficiency is easy to check. The necessity follows from Proposition 2 (in subsection 7.1 of Appendix) and Theorem 1. □

7.4 Proof of Theorem 3

Fix any continuous regular mechanism $(d; \ )$ that satisfies AN, AS, NBD, SP. Given Proposition 1 and Fact 1; the result would follow if we show that the threshold function $T(.)$ associated with $(d; \ )$ is of the following form:

$$\text{(a)} \quad T(v_{-i}) = \max\{v_{-i}(m); \ \}; \forall i \in N; \forall v \in \mathbb{R}^N_+;$$

where, as in proof of Theorem 1, $:= \inf\{x \geq 0 : x^n \notin B^m_0\}$ and, as defined earlier, $B^m_0 = \{v \in \mathbb{R}^N_+ : W^m(v) = \emptyset\}$. We establish (a) in the following four steps:

Step 3

In this step we show that for any z 2 $R^{n \; 1}$ with $z(1) = \; > z\,(n \; 1)$, $T(z) = \;$.[27] Fix any such z. Since Proposition 2 has established $T(:)$ as a symmetric function, we can assume w.l.o.g. that $z_k = z(k)$ for all $k = 1; : : : ; n \; 1$. Fix the number h 2 f $1; : : : ; n \; 1$g such that $z_h \; > z_{h+1}$. For any x 2 $[0; \;]$, de ne

$$z^x := (\underbrace{\; x; \; x; : : : ; \; x}_{h \text{ coordinates}}; z_{h+1}; z_{h+2}; : : : ; z_n):$$

Now, consider the rst possibility that $T(z) < \;$. By continuity of $T(:)$ function, there exists an $\; > 0$ such that for all x 2 $(0; \;$ T by construction, $q^{n\ 2}_{m} = q^{n\ 2}_{m+1} =$ and $q^{n\ 2}_{f\ m g} = q^{n\ 2}_{f\ m+1\ g} = z$. Thus, arguing as in Remark 1, the fact that $d_m(q^{n\ 2}) = 1; d_{m+1}(q^{n\ 2}) = 0$ can be used to infer that $T(q^{n\ 2}_{f\ m g}) = T(q^{n\ 2}_{f\ m+1\ g})$ implying that $T(z) = \ = z_m$.

Case 2: $<$

Fix any x 0, de ne the pro le p^x such that $p^x_1 = x$ and $p^x_{\ 1} = z$ and construct the pro le p such that $p_k = p^x(k)$ for all $k\ 2\ f\ 1; 2; :::; n g$. Therefore, by construction, $p_1 \quad p_2 \quad ::: \quad p_n$. We consider two subcases: x and $x <$. In the following paragraphs, we show that $d_1(p^x) = 1$ in the former case while $d_1(p^x) = 0$ in the latter case. By Fact 1, this inference to imply that $T(p^x_{\ 1}) = T(z) =$.

Subcase 1: x

De ne the agent $g := f\ j\ 2\ N : p_j$ and $p_{j+1} <\ g$. Since $<$ and $x >$, agent g is well de ned and $g\ 2\ f\ 1; :::; m g$. Therefore, p_g is the smallest coordinate of p greater than or equal to while p_{g+1}

7.6 Relation between non bossiness in decision (NBD) and Satterthwaite and Sonnenschein [26] version of non-bossiness (SSNB).

Lemma 4. If a strategyproof mechanism (d, t) violates NBD then it violates SSNB.

Proof: Fix any mechanism (d, t) that violates NBD. Therefore, there exists $i \in N$, $v_{-i} \in \mathbb{R}^{N \setminus \{i\}}$, and $x' \neq y' \geq 0$ such that

$$d_i(x', v_{-i}) = d_i(y', v_{-i}) \text{ and } \exists\, j \in N \setminus \{i\} \text{ such that } d_j(x', v_{-i}) \neq d_j(y', v_{-i}).$$

Now it is well known that in a discrete object allocation problem with unit demand, a strategyproofness mechanism exhibits the property that $d_i(a, v_{-i}) = d_i(b, v_{-i}) \implies t_i(a, v_{-i}) = t_i(b, v_{-i})$ for all a, b

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